Compositional design for time-varying and nonlinear coordination

Jonas Hansson and Emma Tegling

***Abstract*—This work addresses the design of high-order consensus protocols for multi-agent coordination. While first-order consensus strategies are well understood—with robustness guarantees under time delays, time-varying weights, and nonlinearities such as saturation—the theoretical foundations for high-order coordination remain limited. We propose a compositional control framework in which high-order consensus is achieved by cascading stable first-order consensus protocols. Under mild assumptions, we show that the resulting system inherits the stability from its components. The versatility of the design is illustrated through an example inspired by vehicular formation control under time-varying connectivity.**

I. Introduction

The coordination of multi-agent systems has a long history, dating back to early works such as [1]. A modern theory of consensus was developed in the early 2000s through pioneering contributions like [2]–[5].

While linear time-invariant consensus protocols are well understood, the theory of *high-order* coordination remains comparatively underdeveloped. One of the simplest models capable of achieving high-order consensus is

$$x_i^{(n)} = u_i(x,t),$$

where each control input u_i relies solely on local and relative information. A prototypical high-order consensus protocol was proposed in [6]:

$$x^{(n)} = -r_{n-1}Lx^{(n-1)} - r_{n-2}Lx^{(n-2)} - \cdots - r_0 Lx,$$

which can achieve coordination of position and the first $n-1$ derivatives, provided that the gains r_k are appropriately tuned. However, as shown in [7], such protocols may suffer from *scale fragility*: for certain network topologies, it is not possible to choose fixed gains r_k that ensure stability without precise knowledge of the Laplacian spectrum. This highlights the need for more scalable approaches to high-order coordination.

In this work, we propose a novel control design that achieves high-order consensus through a cascade of simpler coordination mechanisms. Our design parameterizes a broad class of closed-loop systems that, under suitable conditions, ensure high-order coordination. Importantly, the resulting controllers rely only on local and relative measurements.

We define the *compositional consensus system* as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{L}_n\right) \circ \cdots \circ \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{L}_1\right)(x) = 0, \tag{1}$$

where each $\mathcal{L}_k$ is a (possibly nonlinear and time-varying) Laplacian-like operator. This structure enables a modular analysis: stability of the full system follows from the properties of its first-order components. The linear version of this framework, which we call the *serial consensus*, has already demonstrated promising performance in [8], [9].

The authors are with the Department of Automatic Control and the ELLIIT Strategic Research Area at Lund University, Lund, Sweden. Email: {jonas.hansson, emma.tegling}@control.lth.se

This work was partially funded by Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation.

II. Problem Setup

We consider a network of N identical n^{th}-order integrators governed by

$$x^{(n)}(t) = u(x,t),$$

where $x(t) \in \mathbb{R}^N$ and $u(x,t) \in \mathbb{R}^N$. Our goal is to achieve high-order coordination using only local and relative feedback, as formalized below.

Definition 1 (n^{th}-order consensus)**.** *A solution $x(t) \in \mathbb{R}^N$ achieves n^{th}-order consensus if*

$$\lim_{t\to\infty} |x_i^{(k)}(t) - x_j^{(k)}(t)| = 0, \quad \forall i \neq j,\ k = 0, \ldots, n-1.$$

To this end, we construct the controller using a sequence of first-order coordination protocols of the form

$$\dot{z} = -\mathcal{L}_k(z,t),$$

where each $\mathcal{L}_k : \mathbb{R}^N \times \mathbb{R}_+ \mapsto \mathbb{R}^N$ may be nonlinear and time-varying. These operators are composed to form the control input

$$u(x,t) = x^{(n)} - \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{L}_n\right) \circ \cdots \circ \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{L}_1\right)(x). \tag{2}$$

The composition is taken with respect to the first argument only, i.e.,

$$(\mathcal{L}_2 \circ \mathcal{L}_1)(x) := \mathcal{L}_2(\mathcal{L}_1(x,t),t).$$

Our design enforces relative feedback, a natural constraint in formation control and distributed systems.

Assumption 1 (Relative feedback)**.** *Each coordination protocol $\mathcal{L}_k$ satisfies*

$$\mathcal{L}_k(z + \mathbb{1}a(t), t) = \mathcal{L}_k(z,t),\ \forall z \in \mathbb{R}^N,\ a(t) \in \mathbb{R},\ \text{and } t \geq 0.$$

This ensures invariance under translation and promotes consensus-seeking behavior.

To analyze the stability of (2), we impose the following technical assumptions on $\mathcal{L}_k$ for $k \leq n-1$.

Assumption 2 (Lipschitz and piecewise continuity)**.** *Each $\mathcal{L}_k(z,t)$ is Lipschitz in z with a uniform constant (independent of t) and is piecewise continuous in t for each fixed z.*

Assumption 3 (Local ISS)**.** *If $\|w(t)\| \leq M_k$ for all $t \geq T_0$, then the perturbed system $\dot{z} = \mathcal{L}_k(z,t) + w(t)$ is input-to-state stable (ISS) with respect to a seminorm $|\!|\!|\cdot|\!|\!|$, i.e.,*

$$|\!|\!|z(t)|\!|\!| \leq \beta_k(|\!|\!|z(T_0)|\!|\!|, t) + \gamma_k\left(\sup_{t\geq T_0} \|w(t)\|\right),$$

for some $\beta_k \in \mathcal{KL}$, $\gamma_k \in \mathcal{K}$, where $|||z||| = 0 \iff z \in$ $\mathrm{span}(\mathbb{1})$.

Assumption 4 (Smoothness)**.** *Each $\mathcal{L}_k \in C^{n-1-k}$ satisfies*

$$\|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\mathcal{L}_k(z,t)\| \leq \alpha_{k,j}\left(\max_{0\leq i\leq j}\|z^{(i)}\|\right),$$

for some $\alpha_{k,j} \in \mathcal{K}$, all $j \leq n-k-1$, and all $t \geq 0$.

For further discussion of these assumptions and their implications, we refer the reader to the full article [10].

III. Main Results

The following result establishes that the composition of first-order consensus protocols, as in (1), yields asymptotic n^{th}-order consensus.

Theorem 1. *Let each operator $\mathcal{L}_k$ satisfy the relative feedback condition in Assumption 1, and assume that the unforced system*

$$\dot{z}_k = \mathcal{L}_k(z_k, t)$$

admits a unique solution for any initial condition $z_k(0)$, with

$$\lim_{t\to\infty}\|z_k(t) - \mathbf{1}a_k(t)\| = 0,$$

for some scalar function $a_k(t)$. Further assume that each $\mathcal{L}_k$, for $k = 1, \ldots, n-1$, satisfies Assumptions 2–4. Then the compositional consensus system (1) *admits a unique solution $x(t)$, and this solution achieves n^{th}-order consensus.*

The proof of this result is provided in [10].

IV. Case Study

First-order consensus protocols possess several desirable properties. For example, [5] shows that the system

$$\dot{z} = -L(t)z,$$

achieves exponential consensus if the time-varying graph $L(t)$ is sufficiently connected over time. If we choose $\mathcal{L}_k(z,t) = L_k(t)z$, and ensure that each $L_k(t)$ is smooth in accordance with Assumption 4, the conditions of Theorem 1 are easily verified.

Example 1. *Consider a second-order compositional consensus protocol where $\mathcal{L}_2(z,t) = \mathcal{L}_1(z,t) = D(t)L$, with L a fixed directed Laplacian and $D(t)$ a continuous, nonnegative, bounded diagonal matrix. If L contains a directed spanning tree and there exist constants $T > 0$, $\delta > 0$ such that*

$$\int_t^{t+T} D_{i,i}(\tau)\,\mathrm{d}\tau > \delta \quad \forall t, \text{ and } i$$

then $x(t)$ achieves second-order consensus.

In the simulation shown in Fig. 1, L represents a directed path graph with a leader, and

$$D_{i,i}(t) = \max\left\{\sin(\omega_i t + \phi_i), 0\right\},$$

where ω_i and ϕ_i are randomly generated. The agents implement a compositional consensus control law

$$u(x,t) = -2D(t)L\dot{x} - \left(D(t) + \dot{D}(t)\right)L(x - d_{\mathrm{ref}}),$$

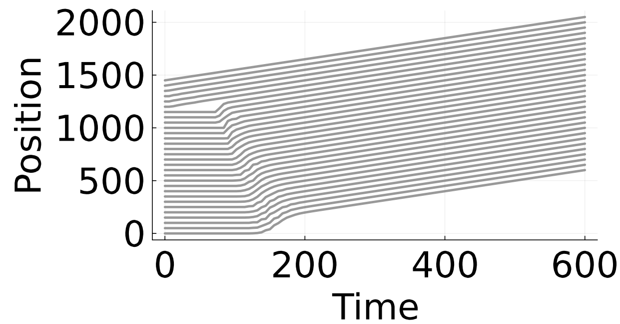

Fig. 1: Second-order compositional consensus under a time-varying coordination protocol. Vehicles return to the desired formation despite initial standstill and intermittent disconnection.

where d_{ref} denotes the desired inter-agent spacing in the formation. This allows each agent to maintain a fixed relative distance from its predecessor while adapting to the time-varying coordination gains. The system successfully achieves second-order consensus, illustrating robustness to time-varying and intermittently disconnected communication.

Further examples, including coordination under input saturation $\mathcal{L}(z,t) = \mathrm{sat}(Lz)$ and time-delayed GPS feedback, are provided in the full article [10].

V. Conclusions

This work introduced the compositional consensus system, a flexible framework for designing high-order coordination protocols using cascades of first-order consensus operators. This structure enables the construction of nonlinear, time-varying, and time-delayed protocols that achieve consensus under mild connectivity assumptions.

By reducing the analysis of complex high-order systems to modular components, the approach provides both theoretical insight and practical scalability. Future work includes extensions to heterogeneous agent dynamics, experimental validation, and distributed observer design for reduced communication overhead.

References

[1] W. Levine and M. Athans, “On the optimal error regulation of a string of moving vehicles,” *IEEE Trans. Autom. Control*, vol. 11, no. 3, pp. 355–361, 1966.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[3] R. Olfati-Saber and R. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[4] J. A. Fax and R. M. Murray, “Information flow and cooperative control of vehicle formations,” *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[5] L. Moreau, “Stability of continuous-time distributed consensus algorithms,” in *Proc. IEEE Conf. Decis. Control*, vol. 4, 2004, pp. 3998–4003.

[6] W. Ren, K. Moore, and Y. Q. Chen, “High-order consensus algorithms in cooperative vehicle systems,” in *Proc. IEEE Int. Conf. Netw. Sens. Control*, 2006, pp. 457–462.

[7] E. Tegling, B. Bamieh, and H. Sandberg, “Scale fragilities in localized consensus dynamics,” *Automatica*, vol. 153, p. 111046, 2023.

[8] J. Hansson and E. Tegling, “Closed-loop design for scalable performance of vehicular formations,” *IEEE Trans. Control Netw. Syst.*, pp. 1–10, 2025.

[9] ——, “Performance bounds for multi-vehicle networks with local integrators,” *IEEE Control Syst. Lett.*, vol. 8, pp. 2901–2906, 2024.

[10] ——, “Compositional design for time-varying and nonlinear coordination,” *Preprint: arXiv:2504.07226*, 2025.