A Unified Framework for the Analysis and Design of Online Feedback Optimization Controllers

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Abstract*—Online feedback optimization (OFO) refers to the design of feedback controllers that guide a physical systems toward the solution of an optimization problem, while enforcing physical and operational constraints. The core idea of this control paradigm is to re-purpose classic optimization algorithms as dynamic feedback controllers by integrating online measurements from the physical plant. Unlike traditional optimization-based controllers, these methods require minimal model knowledge, no external set-points, and low computational effort, while fully leveraging real-time data. Thanks to these advantages, OFO has gained significant traction among researchers and practitioners across diverse fields, including electrical power grids, transportation systems, process control, and robotics. Its effectiveness has been demonstrated through simulations, experimental validation, and even industrial deployment on real distribution grids. In this work, we present a general framework for designing and analyzing OFO controllers, addressing key challenges such as closed-loop stability, robustness, distributed and decentralized designs, and practical implementation aspects. Numerical simulation and hardware experiments in different applications, including swarm robotics, multi-area transmission grids, and recommender systems will be presented to demonstrate the potential and generality of the proposed framework.

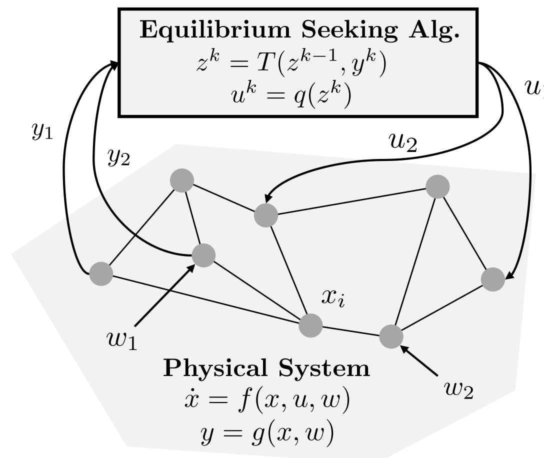

Fig. 1. In feedback equilibrium seeking, measurements from a countinuous-time dynamical system are incorporated into a discrete-time equilibrium seeking algorithm resulting in a coupled sampled-data cyber-physical system.

I. Introduction

Online feedback optimization (FO) [1] is an emerging control paradigm for optimal steady-state operation of complex systems based on their direct closed-loop interconnection with optimization algorithms. FO controllers can handle control objectives beyond set-point regulation, typically tracking (a-priori unknown) solution trajectories of time-varying constrained optimization problems. In recent years, FO controllers have been proposed for a wide variety of problem settings [1]–[7]. These can be categorized by the type of control objective (e.g., convex or nonconvex) and constraints (e.g., hard or soft), the dynamics of the plant (e.g., nonlinear, linear, or algebraic), the type of algorithm (discrete or continuous-time), and the stability analysis (e.g., continuous-time, discrete-time, or hybrid), see [1] for a comprehensive list. FO has found widespread application in various domains, including power systems (e.g., for optimal power reserve dispatch [2], or frequency regulation in AC grids [4]), communication networks (e.g., for network congestion control [6]), and transportation systems (e.g., for ramp metering control [7]).

In this work, we present feedback equilibrium seeking (FES) [8], an extension of FO that seeks to drive pre-stabilized dynamical systems to "efficient" operating points encoded by time-varying generalized equations (GEs). GEs contain constrained optimization as a special case and can model a broad range of equilibrium problems (e.g., Nash, Wardrop).

Our contribution to this area of research is threefold:

(i) We propose a general framework for designing OFO controllers for continuous-time pre-stabilized nonlinear systems by tapping into a broad class of first- and second-order discrete-time algorithms for generalized equations.
(ii) We derive sufficient conditions for stability and robustness of the sampled-data algorithm-plant interconnection.
(iii) We showcase the utility of our framework by means of numerical simulations on smart building [8], multi-area transmission grids [9], recommender systems [10], and via experiments with swarms of Crazyflie quadcopters.

II. Problem Statement

We consider the problem of efficiently operating a physical plant described by the following nonlinear state-space system

$$\dot{x}(t) = f(x(t), u(t), w(t)), \tag{1a}$$
$$y(t) = g(x(t), w(t)) \tag{1b}$$

where $x \in \mathbb{L}^{n_x}$ is the state, $y \in \mathbb{L}^{n_y}$ is the output, $u \in \mathbb{L}^{n_u}$ is the control input, with $u(t) \in \mathcal{U}$ for all $t \in \mathbb{R}_{\geq 0}$, and $w \in \mathbb{L}^{n_w}$ is a disturbance satisfying $w(t) \in \mathcal{W}$ for all $t \in \mathbb{R}_{\geq 0}$.

We adopt the "stabilize-then-optimize" paradigm, and assume that (1) is stable and has a steady-state map $p : \mathcal{U} \times \mathcal{W} \to$

$\mathbb{R}^{n_x}$ satisfying $f(p(u, w), u, w) = 0$ for all $u \in \mathcal{U}, w \in \mathcal{W}$ and a steady-state input-output map

$$h(u, w) = g(p(u, w), w). \quad (2)$$

Our control objective is to design an output feedback controller that drives (1) and maintain it near efficient operating conditions. We encode "efficiency" using the following structured generalized equation (GE), parameterized by $w \in \mathcal{W}$:

$$0 \in G(z, s) + \mathcal{A}(z) \qquad \text{(efficiency objective)} \quad (3a)$$

$$s = h(u, w), \qquad \text{(steady-state map)} \quad (3b)$$

$$u = q(z), \qquad \text{(ctrl state-to-input map)} \quad (3c)$$

where $z \in \mathbb{R}^{n_z}$ is an auxiliary variable, s is the steady-state output of (1), $G : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_z}$ is a single-valued mapping, $\mathcal{A} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_z}$ is a set-valued mapping, and $q : \mathbb{R}^{n_z} \to \mathcal{U}$ is the output mapping.

III. Methodology

Our objective is to maintain the system (1) near efficient operating points, namely, the solution trajectories $s^* = h(u^*, w)$ of the GE in (3). Since (1) is pre-stabilized, selecting $u(t) = u^*(t)$ for all $t \geq 0$ would cause (1) to approximately track the desired steady-state $s^*(t)$. However, computing $u^*(t)$ requires full knowledge of $w(t)$ and evaluating the solution mapping $S(w(t))$ which may be impossible and/or impractical. Instead, we approach the problem by modifying an iterative algorithm for solving the GE (3) with the following form

$$s^k = h(q(z^k), w), \quad (4a)$$

$$z^{k+1} = T(z^k, s^k), \quad (4b)$$

where $T : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_z}$ is the *algorithm*, i.e., a rule for generating the next iterate. This class of algorithms is abstract and broad, including e.g. projected-gradient, SCP, and best-response dynamics in strictly convex games.

By substituting (4a) in (4b), we can compactly cast the algorithmic update rule (4) via the condensed parameterized mapping $\mathbb{T}(\cdot, w) : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$, defined as

$$\mathbb{T}(z, w) := T(z, h(q(z), w)). \quad (5)$$

We assume that the nominal iteration (4) is locally convergent and well-behaved in a parameterized setting.

If $w(t)$ were fully measurable and the steady-state input-output mapping h in (2) perfectly known, then $u^*(t)$ could be computed using (4). Instead, we construct an output feedback controller by replacing the steady-state input-output model s^k in (4a) with online measurements y^k obtained from the physical system (1). This creates an *"online" feedback equilibrium seeking* process, where the system is directly integrated into the algorithm, as illustrated in Figure 1.

IV. Experimental Validation

This design framework has been validated via extensive numerical simulations and real-world experiments. The application explored include real-time coordination of swarms of drones, voltage control in multi-area transmission grids [9], supply-chain systems [8], recommender systems in social networks [10], and temperature control in smart buildings [8]. A representative illustration for this last application is reported in Figure 2.

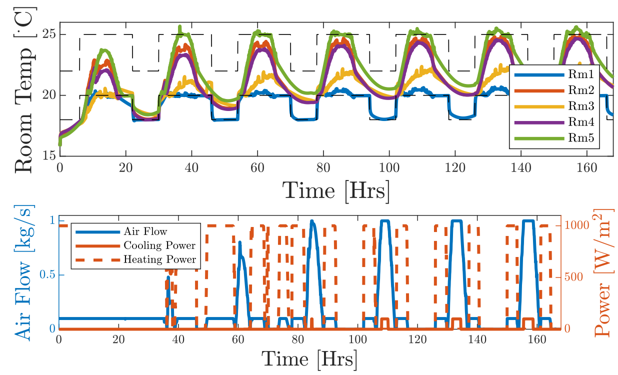

Fig. 2. FES controller for temperature regulation in a smart building. The comfort temperature (output) constraints are (approximately) satisfied throughout the simulations, while heating and cooling effort is minimized.

V. Conclusion

Iterative algorithms for solving generalized equations, such as Josephy–Newton, forward-backward splitting, can be used as sampled-data robust feedback controllers for guiding complex unknown dynamical systems to constrained and economic equilibria. Under robust stability of the plant, strong regularity of the generalized equation describing the control objective, and robust convergence of the iterative algorithm, the sampled-data algorithm-plant cyber-physical interconnection is locally input-to-state stable with respect to unmeasured disturbances, provided that the sampling period is appropriately designed.

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