EV charging: an optimal control approach via PMP

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***Abstract*—This study provides a semi-analytical solution to the charging optimization problem for electric vehicles, using Pontryagin's Maximum Principle (PMP). Explicit expressions for the optimal solution not only enable faster trajectory computation than numerical optimization, but also reveal the solution's structure, increasing interpretability and theoretical insight.**

I. Problem formulation

The vehicle is modeled as a nonlinear system with two states, vector $\mathbf{x}$, and three control inputs, vector $\mathbf{u}$

$$\mathbf{x}(t) = \begin{bmatrix} T_{\mathrm{b}}(t) & \mathrm{SoC}(t) \end{bmatrix}^{\intercal} \tag{1}$$

$$\mathbf{u}(t) = \begin{bmatrix} P_{\mathrm{b}}(t) & P_{\mathrm{hvch}}(t) & P_{\mathrm{hp}}(t) \end{bmatrix}^{\intercal} \tag{2}$$

where T_{b} and SoC are battery temperature and state of charge, respectively. P_{b} is the battery internal chemical power, P_{hvch} is the High Voltage Coolant Heater (HVCH) power, and P_{hp} is the Heat Pump (HP) power. Throughout the article, nested parenthesis have been omitted for increased readability.

The battery pack is modeled as a series circuit between the open circuit voltage U_{oc} and the battery's internal resistance R_{b}, which are modeled as in [1]. The state of charge is the ratio between the available energy in the battery pack and its nominal total energy. Its dynamics are described as

$$\frac{d\mathrm{SoC}(t)}{dt} = -\frac{P_{\mathrm{b}}(t)}{C_{\mathrm{b}} U_{\mathrm{oc}}(\mathrm{SoC})} \tag{3}$$

which means that a negative P_{b} charges the battery.

The heat generated by Joule losses, Q_{joule}, the heat exchanged between the battery and the environment, Q_{amb}, as well as the portion of the HVCH power to heat up the cabin compartment are modeled as in [1]. The heat provided to the battery by the HVCH is modeled as

$$Q_{\mathrm{hvch}}(P_{\mathrm{hvch}}) = \eta_{\mathrm{hvch}} \cdot P_{\mathrm{hvch}}(t) - \alpha_{\mathrm{hvch}} \cdot \left(P_{\mathrm{hvch}}(t)\right)^2 \tag{4}$$

where η_{hvch} is the HVCH efficiency and α_{hvch} is a small coefficient which makes Q_{hvch} quadratic in P_{hvch}. To avoid the HP channeling heat to the cabin when the battery temperature is too low, while still avoiding points of non-differentiability, the heat transfer can be modeled as a smoothed rectified linear unit (ReLU). To that goal, we first define the smoothed step function

$$\sigma(K) = \frac{1}{2}\left(\frac{K}{\sqrt{K^2 + \epsilon^2}} + 1\right) \tag{5}$$

which tends to 1 if $K \geq 0$ and to 0 otherwise. The transition between 0 and 1, when $K \approx 0$, becomes steeper the smaller ϵ is. The heat transferred by the HP can then be modeled as

$$Q_{\mathrm{hp}}(T_{\mathrm{b}}, P_{\mathrm{hp}}) = f_{\mathrm{hp}}\left(T_{\mathrm{b}}, P_{\mathrm{hp}}\right) \cdot \sigma(Tb - T_{\mathrm{b}}^{thres}) \tag{6}$$

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given

$$f_{\mathrm{hp}}\left(T_{\mathrm{b}}, P_{\mathrm{hp}}\right) = \left(p_{\mathrm{Qhp},1} T_{\mathrm{b}} + p_{\mathrm{Qhp},0}\right) P_{\mathrm{hp}} - \alpha_{\mathrm{hp}} P_{\mathrm{hp}}^2 \tag{7}$$

where $p_{\mathrm{Qhp},1}$ and $p_{\mathrm{Qhp},0}$ are coefficients obtained by fitting data, α_{hp} is a small coefficient to make Q_{hp} quadratic in P_{hp} and T_{b}^{thres} is the minimum battery temperature needed to be able to use the HP.

The battery has been modeled as a lumped mass with only the battery temperature as a thermal state, which evolves according to

$$\begin{aligned} \frac{dT_{\mathrm{b}}(t)}{dt} = \frac{1}{c_{\mathrm{b}} m_{\mathrm{b}}} \big(& Q_{\mathrm{joule}}(\mathbf{x}, \mathbf{u}) + Q_{\mathrm{hvch}}(P_{\mathrm{hvch}}) \\ & - Q_{\mathrm{amb}}(T_{\mathrm{b}}) - Q_{\mathrm{hp}}(T_{\mathrm{b}}, P_{\mathrm{hp}}) \big) \end{aligned} \tag{8}$$

where c_{b} is the heat capacity of the battery cells, m_{b} is the mass of the battery pack, and C_{b} is the battery capacity.

We can now define the charging optimization problem, whose goal is to minimize the energy taken from the grid, represented by the stage cost S, and the charging time, represented by the terminal cost V

$$\min_{\mathbf{x},\mathbf{u},t_{\mathrm{f}}} \; J(\mathbf{x}, \mathbf{u}, t_{\mathrm{f}}) = \underbrace{w_{\mathrm{t}} t_{\mathrm{f}}}_{V(t_{\mathrm{f}})} + \int_{t_0}^{t_{\mathrm{f}}} \underbrace{w_{\mathrm{e}} P_{\mathrm{grid}}(\cdot)}_{S(\cdot)} \, dt \tag{9a}$$

$$\text{s.t.: } \mathbf{x}(0) = \mathbf{x}_0, \; \mathrm{SoC}(s_{\mathrm{f}}) = \mathrm{SoC}_{\mathrm{des}} \tag{9b}$$

$$(8), \; (3), \tag{9c}$$

$$P_{\mathrm{b,min}}(\mathbf{x}) \leq P_{\mathrm{b}}(t) \leq 0 \tag{9d}$$

$$x_i \in [x_i^{\min}, x_i^{\max}], \; i = 1, 2 \tag{9e}$$

$$u_i \in [u_i^{\min}, u_i^{\max}], \; i = 1, 2, 3 \tag{9f}$$

where $\tilde{w}_{\mathrm{t}}$ is the penalty for charging time, w_{e} is the penalty for energy bought from the grid, t_0 is the starting time, t_{f} is the final time, P_{grid} is the power taken from the grid, $\mathbf{x}_0 \in \mathbb{R}^2$ are known initial conditions for the states, $\mathrm{SoC}_{\mathrm{des}}$ is the desired SoC level to be reached at the end of charging, $P_{\mathrm{b,min}}$ is a lower bound for P_{b} as a function of the states, $x_i^{\min}$, $x_i^{\max}$, $u_i^{\min}$ and $u_i^{\max}$ are known box constraints on the states and on the control inputs, respectively. Notice how $\mathrm{SoC}(t_{\mathrm{f}})$ is known while $T_{\mathrm{b}}(t_{\mathrm{f}})$ is free.

The trajectories obtained by solving this problem numerically will be used as benchmark and compared with those described in sections (II).

II. Pontryagin's Maximum Principle (PMP)

The Pontryagin's Maximum Principle [2] provides necessary conditions that an optimal trajectory must fulfill. We define two costates (one for each state)

$$\boldsymbol{\lambda}(t) = \begin{bmatrix} \lambda_{T_{\mathrm{b}}}(t) & \lambda_{\mathrm{SoC}}(t) \end{bmatrix}^{\intercal} \tag{10}$$

and the Hamiltonian of the problem in (9), defined according to the theory of the PMP [3]

$$H(\cdot) = S(\cdot) + \lambda_{T_\mathrm{b}}(t)\frac{dT_\mathrm{b}(t)}{dt} + \lambda_{\mathrm{SoC}}(t)\frac{d\mathrm{SoC}(t)}{dt}. \tag{11}$$

The optimality conditions can now be stated as

$$H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \leq H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t),\ \forall t \in [t_0, t_f], \tag{12a}$$

$$\frac{d\boldsymbol{\lambda}}{dt} = -\nabla_x H(\cdot), \tag{12b}$$

$$\lambda_{T_\mathrm{b}}(t_\mathrm{f}) - \frac{\partial V^*}{\partial x} = 0,\ H^*(t_\mathrm{f}) + \frac{\partial V^*}{\partial t_\mathrm{f}} = 0. \tag{12c}$$

where (12a) means that solving (9) is the same as minimizing H, (12b) provides the dynamics of the costates and (12c) are final conditions from PMP theory which hold because both t_f and $T_\mathrm{b}(t_\mathrm{f})$ are free.

The optimal trajectory for the P_b can be found as the trajectory that fulfills $\frac{\partial H}{\partial P_\mathrm{b}} = 0$, since the goal is to minimize H. Solving the equation for P_b, we get the optimal unconstrained battery power

$$P^*_{\mathrm{b,uc}}(\mathbf{x}, \boldsymbol{\lambda}) = \frac{c_\mathrm{b} m_\mathrm{b} \lambda_{\mathrm{SoC}}(t) U_\mathrm{oc}(\mathrm{SoC})}{2C_\mathrm{b}\left(\lambda_{T_\mathrm{b}}(t) + c_\mathrm{b} m_\mathrm{b} w_\mathrm{e}\right) R_\mathrm{b}(T_\mathrm{b})}. \tag{13}$$

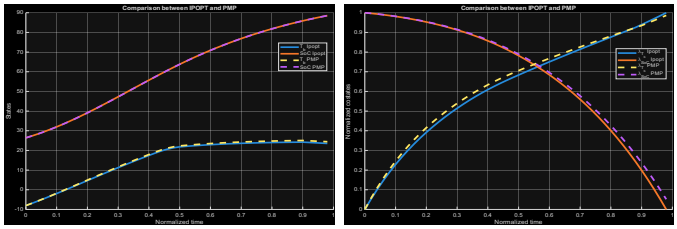
Analogously, the optimal trajectory for P_hvch can be found as the trajectory that fulfills $\frac{\partial H}{\partial P_\mathrm{hvch}} = 0$, which can be solved for P_hvch, since (4) is quadratic in P_hvch, obtaining the optimal unconstrained HVCH power

$$P^*_{\mathrm{hvch,uc}}(\lambda_{T_\mathrm{b}}) = \frac{\eta_\mathrm{hvch}\lambda_{T_\mathrm{b}}(t) + c_\mathrm{b} m_\mathrm{b} w_\mathrm{e}}{2\alpha_\mathrm{hvch}\lambda_{T_\mathrm{b}}(t)}. \tag{14}$$

Finally, the procedure to obtain an expression for the HP power is analogous to the ones for the battery power and the HVCH power, since P_hp enters H quadratically, thanks to Q_hp being quadratic in P_hp. However, this is not enough, as $\frac{\partial^2 H}{\partial P_{hp}^2}$ is not always non-negative, meaning that simply solving $\frac{\partial H}{\partial P_{hp}} = 0$ does not necessarily lead to a minimum of H. Instead, we need to also enforce $\frac{\partial^2 H}{\partial P_\mathrm{hp}^2} \geq 0$. By doing that, we obtain an expression for the optimal unbounded HP power

$$\begin{aligned} P^*_{\mathrm{hp,ub}} = &\frac{1}{2\sigma_\mathrm{hp}(T_\mathrm{b})\left(\frac{\alpha_\mathrm{hp}\, w_\mathrm{e}}{\eta_\mathrm{Qhvch}} + \frac{\alpha_\mathrm{hp}\,\lambda_{T_\mathrm{b}}}{c_\mathrm{b}\, m_\mathrm{b}}\right)} \cdot \\ &\cdot \left[w_\mathrm{e}\left(\frac{\sigma_\mathrm{hp}(T_\mathrm{b})\left(Q_{\mathrm{hp,P0}} + Q_{\mathrm{hp,P1}}\, T_\mathrm{b}\right)}{\eta_\mathrm{Qhvch}} - 1\right) + \right. \\ &\left. + \frac{\lambda_{T_\mathrm{b}}\,\sigma_\mathrm{hp}(T_\mathrm{b})\left(Q_{\mathrm{hp,P0}} + Q_{\mathrm{hp,P1}}\, T_\mathrm{b}\right)}{c_\mathrm{b}\, m_\mathrm{b}} \right]. \end{aligned} \tag{15}$$

At any time instant, the control inputs are chosen so that H is minimized. For P_hvch and P_hp, three cases are considered: unconstrained optimum ($P^*_{\mathrm{hvch,uc}}$ and $P^*_{\mathrm{hp,uc}}$ respectively), lower bound and upper bound (in case the unconstrained optimum goes outside the bounds). For P_b, only the unconstrained optimum $P^*_{\mathrm{b,uc}}$ and the lower bound $P_{\mathrm{b,min}}$ are considered, since the upper bound would be either 0 or positive, which should not happen during charging.



(a) States. (b) Normalized costates.

Fig. 1. Comparison between IPOPT (continuous) and PMP (dashed)

The expressions for the unconstrained optimal control inputs depend on the co-states, whose initial values we do not have. We also need to compute the charging time to be able to simulate the system. The unknowns to be found are therefore $\boldsymbol{\omega} = \begin{bmatrix}\lambda_{T_\mathrm{b}}(0) & \lambda_{\mathrm{SoC}}(0) & t_\mathrm{f}\end{bmatrix}^\intercal$. To compute these three unknowns, a two-point boundary value problem defined by (9b) and (12c) is solved numerically through Newton method, starting from an initial guess and correcting it until convergence. This 2PBVP is the only part that is solved online, since the expressions for the optimal control inputs are computed offline only once.

III. Results and Conclusions

Problem (9) was solved with IPOPT [4] through CasADi [5] to obtain benchmark trajectories. Fig.1 shows an example run where the states and the costates obtained with the PMP-based method are compared with the benchmark ones. It can be seen how the trajectories obtained with the two methods are nearly identical. Table I shows some numerical results for the same run, where it can be seen that the 2PBVP has been solved correctly with the PMP-based method in significantly less time than it took to solve the problem with IPOPT. This shows that the PMP-based method not only is of theoretical interest, since it uncovers the structure of the optimal solution, but it also allows a much faster retrieval of said solution.

TABLE I
Results comparison between IPOPT and PMP

	Exec time [ms]	t_f [min]	$\lambda_{T_\mathrm{b}}(0)$	$\lambda_{\mathrm{SoC}}(0)$
IPOPT	69.86	46.04	-3.3	-165.27
PMP	0.18	45.71	-3.51	-173.77

References

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