Opinion optimization and the wisdom of crowds in social networks

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***Abstract*— We investigate how to optimize the wisdom of a group by modifying individuals susceptibilities to social influence. Departing from existing approaches that minimize the mean square error of collective estimates, we adopt the absolute error as the performance metric for group wisdom. We formally formulate the optimization problem and show that it can be cast as a quadratic program (QP), which admits a relaxation to a QP over the probability simplex. Solving this problem yields an optimal allocation of social power that minimizes the groups absolute error. We further demonstrate that the optimal susceptibilities can be approximated using the pseudo-inverse of the network Laplacian. In the special case of a strongly connected influence network, we derive an explicit form for the optimal susceptibilities.**

I. Introduction

Empirical experiments suggest that simply pooling many independent opinions on an unknown truth yields collective opinions essentially close to the truth, known as the effect of the wisdom of crowds [2]. A social influence process is a natural system that aggregates individuals' respective opinions through the interaction among them. It has long been debated that whether social influence improves the wisdom of crowds or undermines it [1], [4]. In [5], theoretical results are provided to explain how social influence improves, undermines and optimizes the wisdom of crowds based on the influence system theory.

In this paper, we study the problem of how to optimize the wisdom of a group by modifying individuals' susceptibilities. Departing from existing work that focuses on the mean square error of collective estimates, we adopt the absolute error as our measure of collective wisdom. We propose formal formulation of the problem and show that it can be recast as a QP, which can be further relaxed to a QP over the simplex. By solving the QP, we obtain the optimal social power allocation for optimizing the wisdom. The corresponding optimal susceptibilities can be approximated using the pseudo-inverse of the network Laplacian. Specifically, if the influence network is strongly connected, we derive an explicit solution for the optimal susceptibilities.

II. Problem formulation

First, we introduce the naïve learning setting in social networks [3]. Consider n individuals interacting their opinions for an unknown state of the nature with truth μ in an influence network. At the beginning, each individual independently proposes its initial opinion $y_i(0)$, which is a random distribution with expectation μ_i. Then, individuals evolve their opinions on the unknown truth in an influence network $\mathcal{G}(W)$ according to the FJ opinion dynamics:

$$y_i(k+1) = a_i \sum_{j=1}^{n} W_{ij} y_j(k) + (1-a_i) y_i(0) \qquad (1)$$

where $a_i(s)$ is i's susceptibility and W is the row-stochastic adjacent matrix of the influence network. Let $\boldsymbol{a} = (a_1, \dots, a_n)^\top$. Suppose $\boldsymbol{a} < \mathbf{1}_n$ and $\boldsymbol{a} \neq \mathbf{0}_n$, i.e., all individuals are stubborn but are not all fully stubborn, we have $\boldsymbol{y}(\infty) = V\boldsymbol{y}(0)$ with

$$V = (I_n - \operatorname{diag}(\boldsymbol{a})W)^{-1}(I_n - \operatorname{diag}(\boldsymbol{a})). \qquad (2)$$

Denote by $\boldsymbol{y}_{\text{ave}} = (1/n)\sum_{i=1}^{n} y_i(\infty)$ the collective estimate, then $\boldsymbol{y}_{\text{ave}} = \boldsymbol{x}^\top \boldsymbol{y}(0)$, where $\boldsymbol{x} = (1/n)V^\top \mathbf{1}_n$ is the social power allocation of the FJ model and measures the relative control of individual initial opinions on others' final opinions. As a result, we obtain

$$\mathbb{E}[\boldsymbol{y}_{\text{ave}}] = \sum_{i=1}^{n} x_i \mu_i.$$

The difference between the collective estimate y_{ave} and the truth μ measures the deviation of the collective estimate from the truth, thereby indicates the level of collective intelligence and the performance of the influence system (1). Define $f(\boldsymbol{a}) : \mathbb{R}^n \to \mathbb{R}$ by $f(\boldsymbol{a}) = |\mathbb{E}[\boldsymbol{y}_{\text{ave}}] - \mu|$.

Given influence network $\mathcal{G}(W)$, the truth μ and individual expertise indicated by μ_i, assume that we have access to manipulate individual susceptibilities a_i, our objective is to shape individual susceptibilities a_i to minimize the absolute error $f(\boldsymbol{a})$. Formally,

$$\begin{aligned} &\textbf{minimize} \quad f(\boldsymbol{a}) \\ &\textbf{subject to} \quad \boldsymbol{x} = (I_n - \operatorname{diag}(\boldsymbol{a}))(I_n - W^\top \operatorname{diag}(\boldsymbol{a}))^{-1}\frac{\mathbf{1}_n}{n}, \\ &\textbf{variables}: \quad \boldsymbol{a} = (a_1, \dots, a_n)^\top \in [0,1)^n. \end{aligned} \qquad (3)$$

Notably, the objective function $f(\boldsymbol{a})$ is neither convex nor concave.

III. Main results

A. Problem reformulation

Let $\delta_i = \mu_i - \mu$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)^\top$, then $f(\boldsymbol{a}) = |\boldsymbol{x}^\top \boldsymbol{\delta}|$. For the social power $\boldsymbol{x}$, we have the following lemma.

Lemma 1 *For the social power $\boldsymbol{x}$, define $F(\boldsymbol{a}) : \mathbb{R} \to \mathbb{R}^{n\times n}$*

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by

$$F(\boldsymbol{a}) = \mathbf{1}_n\mathbf{1}_n^\top \frac{1}{n} - \operatorname{diag}(\boldsymbol{a})(I_n - \operatorname{diag}(\boldsymbol{a}))^{-1}(I_n - W).$$

Then for all $\boldsymbol{a} \in [0,1)^n$*,*

(i) $F(\boldsymbol{a})$ *is irreducible;*
(ii) $\rho(F(\boldsymbol{a})) = 1$*;*
(iii) $\boldsymbol{x}$ *is the dominant left eigenvector of* $F(\boldsymbol{a})$ *associated with eigenvalue* 1*, i.e.,* $\boldsymbol{x}^\top F(\boldsymbol{a}) = \boldsymbol{x}^\top$*.*

Based on Lemma 1, we can reformulate problem (3) as follows. Denote by $\mathcal{D} = [0,1)^n$. Note that

$$\arg\min_{\boldsymbol{a}\in\mathcal{D}} |\boldsymbol{x}^\top\boldsymbol{\delta}| = \arg\min_{\boldsymbol{a}\in\mathcal{D}} (\boldsymbol{x}^\top\boldsymbol{\delta})^2. \quad (4)$$

Define $g(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ by $g(\boldsymbol{x}) = (\boldsymbol{x}^\top\boldsymbol{\delta})^2 = \boldsymbol{x}^\top\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{x}$.

Theorem 1 *Let* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)^\top$ *with* $\gamma_i = a_i/(1-a_i)$*. Then, solving problem* (3) *is equivalent to solve*

$$\begin{aligned} &\textbf{minimize} \quad g(\boldsymbol{x}) \\ &\textbf{subject to} \quad (\mathbf{1}_n\mathbf{1}_n^\top \frac{1}{n} - (I_n - W^\top)\operatorname{diag}(\boldsymbol{\gamma}))^{-1})\boldsymbol{x} = \boldsymbol{x}, \\ &\qquad\qquad\quad \boldsymbol{x}^\top\mathbf{1}_n = 1, \\ &\textbf{variables}: \quad \boldsymbol{x}, \boldsymbol{\gamma} \in [0,\infty)^n. \end{aligned} \quad (5)$$

B. Optimal social power

Since $\boldsymbol{\delta}\boldsymbol{\delta}^\top$ is positive semi-define, motivated by (4), we relax (5) as

$$\begin{aligned} &\textbf{minimize} \quad \boldsymbol{x}^\top\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{x} \\ &\textbf{subject to} \quad \boldsymbol{x}^\top\mathbf{1}_n = 1, \\ &\textbf{variables}: \quad \boldsymbol{x} \geq \mathbf{0}_n, \end{aligned} \quad (6)$$

which is a quadratic programming (QP) over the simplex. Since $\boldsymbol{\delta}\boldsymbol{\delta}^\top$ is positive semi-define for any $\boldsymbol{\delta} \in \mathbb{R}^n$, problem (6) has optimal solutions. Intuitively, if the components of $\boldsymbol{\delta}$ are all non-negative or non-positive, (6) is solved by $\boldsymbol{x}^* = \boldsymbol{e}_r$, where $r = \arg\min_i \mid \boldsymbol{\delta}_i \mid$. That is, if all experts either overestimate or underestimate the truth, assigning all social power to one expert who has the best estimate optimizes the wisdom.

Otherwise, (6) is more nuanced. We introduce the Lagrange multiplier λ and the Lagrange function

$$\mathcal{L}(\boldsymbol{x}, \lambda) = \boldsymbol{x}^\top\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{x} + \lambda(\boldsymbol{x}^\top\mathbf{1}_n - 1).$$

By the KKT conditions, the optimal solution $\boldsymbol{x}^*$ satisfies

$$2\boldsymbol{\delta}^\top\boldsymbol{x}^*\boldsymbol{\delta}_i + \lambda = 0$$

for all i. Since the components of $\boldsymbol{\delta}$ are not uniform, this implies $\boldsymbol{\delta}^\top\boldsymbol{x}^* = 0$. Therefore, (6) is equivalent to find the vector in the simplex that is orthogonal to $\boldsymbol{\delta}$.

C. Optimal susceptibilities

By solving problem (6), we obtain the optimal social power $\boldsymbol{x}^*$ that minimizes $h(\boldsymbol{x}) = \boldsymbol{x}^\top\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{x}$. Now, we shall solve the corresponding distribution of susceptibilities $\boldsymbol{a}^*$ which yields $\boldsymbol{x}^*$. First, we propose a result for the case that all experts' estimates are systematically biased, i.e., are consistently either larger or smaller than the truth.

Theorem 2 *Suppose that either* $\boldsymbol{\delta} \geq 0$ *or* $\boldsymbol{\delta} \leq 0$*. Let* $r = \arg\min_i \mid \boldsymbol{\delta}_i \mid$*. Then, the original problem* (3) *is solved by* $\boldsymbol{a}^*$ *satisfying* $\boldsymbol{a}_r^* < 1$ *and* $\boldsymbol{a}_i^* = 1$ *for all* $i \neq r$*.*

Theorem 2 means that if all estimates are systematically biased, the wisdom is optimized if only the expert who has the smallest bias is resistant to social influence. Next, we consider the case that $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$.

Lemma 2 *Suppose that* $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ *is the solution of problem* (6)*, then the solution* $\boldsymbol{a}^*$ *of problem* (3) *with* $\gamma_i^* = a_i^*/(1-a_i^*)$ *satisfies*

$$(I_n - W^\top)\operatorname{diag}(\boldsymbol{x}^*)\boldsymbol{\gamma}^* = \frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*. \quad (7)$$

Thus,

$$\boldsymbol{\gamma}^* \approx \operatorname{diag}^{-1}(\boldsymbol{x}^*)(L^\dagger)^\top(\frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*),$$

where $L^\dagger$ *is the pseudo-inverse of* $L = I_n - W$*.*

In Lemma 2, we exploit the pseudo-inverse of the Laplacian $L = I_n - W$, which is equivalent to solve the following least-square problem:

$$\begin{aligned} &\textbf{minimize} \quad \| (I_n - W^\top)\operatorname{diag}(\boldsymbol{x}^*)\boldsymbol{\gamma} - (\frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*) \|^2 \\ &\textbf{variables}: \quad \boldsymbol{\gamma} \geq \mathbf{0}_n. \end{aligned} \quad (8)$$

However, in specific cases, we can guarantee that equations (7) has at least one solution.

Theorem 3 *Suppose that the influence matrix* W *is primitive, i.e., irreducible and aperiodic, and* $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ *is the solution of problem* (6)*. Then, problem* (3) *is solved by*

$$\boldsymbol{\gamma}^* = \operatorname{diag}^{-1}(\boldsymbol{x}^*)(L^\top)^\dagger(\frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*) + \beta\operatorname{diag}^{-1}(\boldsymbol{x}^*)\boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ *is the dominant left eigenvector of* W*,* $\beta \in \mathbb{R}$ *is any constant.*

Example 1 *Consider an influence network consisting of* 3 *experts with adjacency matrix*

$$W = \begin{bmatrix} 0.2 & 0 & 0.8 \\ 0.4 & 0.3 & 0.3 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$

Assume that the truth is 5*, and the expectations of individuals' estimates are* $3, 10, 8$*, respectively. We solve problem* (3) *using the* CVX *package with* MATLAB*. By solving problem* (6)*, we obtain* $\boldsymbol{x}^* = (0.4936, 0.2276, 0.2788)^\top$*. Consequently, exploiting Lemma 2, we have* $\boldsymbol{a}^* = (0.1203, 0.7018, 0.6589)^\top$*. In this case,* $\boldsymbol{\delta} = (2, -5, -3)^\top$ *and* $\boldsymbol{\delta}^\top\boldsymbol{x}^* = 0$*, i.e., they are orthogonal to each other.*

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