Extended Abstract

Free-Spacing Circular Motion of Vehicles in Cyclic Pursuit

Zhaozhan Yao, Yuhua Yao, Xiaoming Hu

Department of Mathematics, KTH Royal Institute of Technology, Stockholm 100 44, Sweden
E-mail: zhaozhan@kth.se, yuhuay@kth.se, hu@kth.se

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1 Introduction

The cyclic pursuit problem, which dates back to 1877, has attracted the interest of mathematicians for over a century. In cyclic pursuit, n bugs chase one another in cyclic order. Since the early 2000s, there has been renewed interest in studying cyclic pursuit from a control perspective, particularly for decentralized control of autonomous agents. In [1], Marshall *et al.* consider nonholonomic vehicles (modeled as kinematic unicycles) under cyclic pursuit, each traveling at a fixed common forward speed. By employing a steering law where each vehicle's angular velocity is proportional to its bearing relative to its prey, the vehicles eventually converge to equally spaced circular motion. Depending on their bearing spacing (on the circle), there are $2n-1$ equilibrium formations. Local stability analysis of these equilibrium formations is presented, revealing which formations are stable and which are not. Following the studies in [1], Zheng *et al.* [2] proposed a new control law to ensure certain collective behaviors. It is guaranteed that only two equilibrium formations are locally asymptotically stable, where the vehicles are equally spaced on a circle in cyclic order, either clockwise or counterclockwise.

Most existing research has shifted its focus toward achieving a desired formation on the circle. The desired formation includes specific bearing spacing, the direction of circular motion, and optionally the circle's center, radius, or both. Additionally, the topology is no longer limited to cyclic pursuit but extends to more general topologies, including directed graphs. In [3], a dynamic control law is developed such that the vehicles converge to a desired circular motion, prescribed by a bearing spacing, a stationary center, and a radius. Global convergence is guaranteed, and the center is the root of the topology. Similar results within the cyclic pursuit framework are developed in [4]. In this work, however, the agent dynamics are described by natural Frenet frame equations rather than kinematic unicycles.

To go beyond prior studies that focus on equally spaced or prescribed-spacing circular motion, we aim to achieve a more general form of circular motion–free-spacing circular motion, where the bearing spacing is not constrained. We first establish the geometric constraints for vehicles moving on a common circle and then propose a control law that enables free-spacing circular motion. A local stability analysis is provided for the case $n \leq 3$, offering insights into the stability properties of the more general case with $n > 3$.

2 Cyclic Pursuit Model

Consider cyclic pursuit of n vehicles, where vehicle $i \in \mathcal{I} = \{1, 2, \dots, n\}$ pursues the next $i+1$ vehicle. The vehicles are modeled as kinematic unicycles: $\dot{x}_i = v\cos\theta_i$, $\dot{y}_i = v\sin\theta_i$, $\dot{\theta}_i = \omega_i$, in which $v > 0$ is a fixed forward speed, $z_i = [x_i, y_i]^\top \in \mathbb{R}^2$ denotes the position of vehicle i, $\theta_i \in \mathbb{R}$ the orientation, and $\omega_i \in \mathbb{R}$ the angular velocity to be designed.

The vehicles only have access to relative measurements, which motivates the use of relative coordinates. Let ρ_i denote the distance between i and its prey $i+1$, α_i denote the bearing from i's heading to the heading that would take it directly towards its prey, and $\beta_i := \theta_i - \theta_{i+1} - \pi$ denote the heading difference between i and its prey minus π. After algebraic manipulations, the motion equations of these relative variables are obtained as:

$$\dot{\rho}_i = -v\left(\cos\alpha_i + \cos\left(\alpha_i + \beta_i\right)\right), \tag{1a}$$

$$\dot{\alpha}_i = \frac{v}{\rho_i}\left(\sin\alpha_i + \sin\left(\alpha_i + \beta_i\right)\right) - \omega_i, \tag{1b}$$

$$\dot{\beta}_i = \omega_i - \omega_{i+1}. \tag{1c}$$

Denote $\xi_i = [\rho_i, \alpha_i, \beta_i]^\top$ and $\xi = [\xi_1^\top, \xi_2^\top, \dots, \xi_n^\top]^\top$. Note that $\sum_{i=1}^n z_{i+1} - z_i = 0$. By choosing a coordinate frame attached to vehicle 1 and oriented with vehicle 1's heading, this equality gives rise to the following coordinate constraints:

$$\begin{aligned} g_1(\xi) &:= \rho_1\sin\alpha_1 + \rho_2\sin\left(\alpha_2 + \pi - \beta_1\right) + \dots \\ &+ \rho_n\sin\left(\alpha_n + (n-1)\pi - \beta_1 - \beta_2 - \dots - \beta_{n-1}\right) = 0, \\ g_2(\xi) &:= \rho_1\cos\alpha_1 + \rho_2\cos\left(\alpha_2 + \pi - \beta_1\right) + \dots \\ &+ \rho_n\cos\left(\alpha_n + (n-1)\pi - \beta_1 - \beta_2 - \dots - \beta_{n-1}\right) = 0. \end{aligned}$$

On the other hand, due to cyclic pursuit $\sum_{i=1}^n \dot{\beta}_i(t) = 0$, it follows $\sum_{i=1}^n \beta_i(t) \equiv c$, where the constant $c = -n\pi$ by the definition for β_i, which gives a final coordinate constraint: $g_3(\xi) := \sum_{i=1}^n \beta_i + n\pi = 0 \bmod 2\pi$. These constraints are essential for equilibrium stability analysis.

Definition 1 *Given a team of vehicles in cyclic pursuit described by* (1). *We say that the vehicles perform a coordinated circular motion if they move on a common circle.*

Definition 2 *An arrangement refers to the relative ordering of the vehicles on the circle.*

Definition 3 *An arrangement is said to be regular if the vehicles are* distinctly *spaced along the circle in cyclic order, following the direction of motion.*

Proposition 1 *A coordinated circular motion occurs if and only if: (i) It holds that $\sum_{i=1}^{n}\beta_i + n\pi = 0$ mod 2π; (ii) It holds that $\alpha_i + (\alpha_i+\beta_i) = \pi$ mod 2π, $\forall i \in \mathcal{I}$; (iii) $2\sin\alpha_i/\rho_i = 1/r \neq 0$, $\forall i \in \mathcal{I}$. $|r|$ is the radius of the circle. $\bar{\omega} = v/r$ is the identical angular velocity of the vehicles. The direction of the motion is indicated by the sign of r. It is counterclockwise if $r > 0$; otherwise, it is clockwise.*

To achieve coordinated circular motion, we propose the following control law with $k \neq 0$:

$$\omega_i = \frac{v}{\rho_i} 2\sin\alpha_i + \frac{k}{\rho_i}\left(\cos\alpha_i + \cos\left(\alpha_i+\beta_i\right)\right). \quad (2)$$

Substituting (2) into (1) yields a cyclically interconnected system of identical nonlinear subsystems, which are omitted for brevity. View these subsystems as $\dot{\xi}_i = f(\xi_i, \xi_{i+1})$ and view the complete system as $\dot{\xi} = \hat{f}(\xi)$. We discuss the possible equilibrium formations of $\dot{\xi} = \hat{f}(\xi)$ in the following theorem, where $\bar{\xi} = [\bar{\xi}_1^\top, \bar{\xi}_2^\top, \ldots, \bar{\xi}_n^\top]^\top$ with $\bar{\xi}_i = [\bar{\rho}_i, \bar{\alpha}_i, \bar{\beta}_i]^\top$ represents an equilibrium point.

Theorem 1 *The equilibria of systems* (1) *with the control* (2) *can be categorized into two formations,*
(i) the first formation $\mathcal{M}_\mathsf{l} \subset \mathbb{R}^{3n}$ is given by

$$\mathcal{M}_\mathsf{l} = \left\{\bar{\xi} \in \mathbb{R}^{3n} \,\middle|\, \sum_{i=1}^{n} \bar{\rho}_i \cos\bar{\alpha}_i = 0\right\} \cap \quad (3)$$
$$\left\{\bar{\xi} \in \mathbb{R}^{3n} | \bar{\alpha}_i = 0 \text{ mod } \pi,\ \bar{\beta}_i + \pi = 0 \text{ mod } 2\pi,\ \forall i \in \mathcal{I}\right\},$$

such equilibria correspond to collinear motion, i.e., the vehicles advance on a straight line;
(ii) the second one $\mathcal{M}_\mathsf{c} \subset \mathbb{R}^{3n}$ is given by

$$\mathcal{M}_\mathsf{c} = \left\{\bar{\xi} \in \mathbb{R}^{3n} \,\middle|\, \sum_{i=1}^{n} \bar{\beta}_i + n\pi = 0 \text{ mod } 2\pi\right\} \cap \quad (4)$$
$$\left\{\bar{\xi} \in \mathbb{R}^{3n} \,\middle|\, 2\bar{\alpha}_i + \bar{\beta}_i = \pi \text{ mod } 2\pi,\ \frac{\sin\bar{\alpha}_i}{\bar{\rho}_i} = \bar{s} \neq 0,\ \forall i \in \mathcal{I}\right\},$$

such equilibria correspond to coordinated circular motion, i.e., the vehicles have identical angular velocity of $\bar{\omega} = 2v\bar{s}$.

Free-spacing circular motion is feasible under the equilibrium formation $\mathcal{M}_c$, as $\bar{\alpha}_i$ is not fixed.

Denote $\tilde{\xi}_i = \xi_i - \bar{\xi}_i$, then linearizing the system about an equilibrium point $\bar{\xi}$ gives rise to n identical linear subsystems of the form $\dot{\tilde{\xi}}_i = A_i\tilde{\xi}_i + B_{i+1}\tilde{\xi}_{i+1}$, where

$$A_i = \begin{bmatrix} 0 & 2v\bar{s}_i\bar{\rho}_i & v\bar{s}_i\bar{\rho}_i \\ 0 & 2k\bar{s}_i - 2v\bar{c}_i & k\bar{s}_i - v\bar{c}_i \\ -2v\bar{s}_i/\bar{\rho}_i & 2v\bar{c}_i - 2k\bar{s}_i & -k\bar{s}_i \end{bmatrix},$$
$$B_{i+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2v\bar{s}_{i+1}/\bar{\rho}_{i+1} & 2k\bar{s}_{i+1} - 2v\bar{c}_{i+1} & k\bar{s}_{i+1} \end{bmatrix},$$
$$\bar{s}_i = \sin\bar{\alpha}_i/\bar{\rho}_i,\ \bar{c}_i = \cos\bar{\alpha}_i/\bar{\rho}_i.$$

Denote $\tilde{\xi} = \xi - \bar{\xi}$, then the complete linearized system has the form $\dot{\tilde{\xi}} = \hat{A}\tilde{\xi}$, where

$$\hat{A} = \begin{bmatrix} A_1 & B_2 & & & \\ & A_2 & B_3 & & \\ & & \ddots & \ddots & \\ & & & A_{n-1} & B_n \\ B_1 & & & & A_n \end{bmatrix}. \quad (5)$$

3 Stability Analysis

By studying the kernel space of the matrix $\hat{A}$, we can directly draw a conclusion on the stability of the equilibrium points in the set $\mathcal{M}_\mathsf{l}$.

Theorem 2 *Every equilibrium point $\bar{\xi} \in \mathcal{M}_\mathsf{l}$ are unstable.*

Lemma 1 *The matrix $\hat{A}$ has at least $n+1$ zero eigenvalues w.r.t. the equilibrium formation $\mathcal{M}_\mathsf{c}$. These eigenvalues can be disregarded when determining the stability of $\hat{A}$ w.r.t. $\mathcal{M}_\mathsf{c}$.*

On the other hand, let $g(\xi) = [g_1(\xi), g_2(\xi), g_3(\xi)]^\top$, then $\mathcal{M} = \{\xi \in \mathbb{R}^{3n} | g(\xi) = 0\} \subset \mathbb{R}^{3n}$ defines a manifold in $\mathbb{R}^{3n}$.

Lemma 2 *The manifold $\mathcal{M}$ is invariant under $\hat{f}$.*

Then, since the manifold $\mathcal{M}$ is invariant under $\hat{f}$, the tangent space $T_{\bar{\xi}}\mathcal{M}$ at every equilibrium point $\bar{\xi} \in \mathcal{M}$ is invariant under $\hat{f}$. Therefore, there exists a change of coordinates for $\mathbb{R}^{3n}$ that transforms $\hat{A}$ into block upper-triangular form

$$\begin{bmatrix} \hat{A}_{T_{\bar{\xi}}\mathcal{M}} & * \\ 0_{3\times(3n-3)} & \hat{A}^\star_{T_{\bar{\xi}}\mathcal{M}} \end{bmatrix}.$$

Lemma 3 *In the quotient space $\mathbb{R}^{3n}/T_{\bar{\xi}}\mathcal{M}$, the induced linear transformation $\hat{A}^\star_{T_{\bar{\xi}}\mathcal{M}} : \mathbb{R}^{3n}/T_{\bar{\xi}}\mathcal{M} \to \mathbb{R}^{3n}/T_{\bar{\xi}}\mathcal{M}$ has solely imaginary eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm j2v\bar{s}$ w.r.t. the equilibrium formation $\mathcal{M}_\mathsf{c}$.*

Thus, we conclude that the matrix $\hat{A}$ w.r.t. $\mathcal{M}_\mathsf{c}$ has at least $n+3$ imaginary-axis eigenvalues that can be disregarded. Thus, the equilibrium points in $\mathcal{M}_\mathsf{c}$ are locally asymptotically stable if the remaining eigenvalues have negative real parts.

Theorem 3 *Consider a equilibrium formation $\mathcal{M}_\mathsf{c}$ with $n \leq 3$, an equilibrium point $\bar{\xi} \in \mathcal{M}_\mathsf{c}$ that corresponds to a regular arrangement is locally asymptotically stable if $k\bar{s} < 0$.*

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