H_2 and H_∞ Optimal Control of Mass-Spring Systems

Johan Lindberg

1 The Problem of Interest

We consider the control of a mass-spring system represented by a transfer function $G(s)$. The goal is to design a controller $K(s)$ that minimizes the effect of external disturbances $w = [w_\mathrm{u}^\mathsf{T}, w_\mathrm{y}^\mathsf{T}]^\mathsf{T}$ on performance outputs $z = [y_\mathrm{G}^\mathsf{T}, u^\mathsf{T}]^\mathsf{T}$. The generalized plant representation is:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \quad u = K(s)y. \tag{1}$$

The closed-loop transfer function from disturbances to performance outputs is given by:

$$T_{zw}(s) = G_{zw}(s) + G_{zu}(s)K(s)(I - G_{yu}(s)K(s))^{-1}G_{yw}(s). \tag{2}$$

The optimal controller minimizes the H_2 or H_∞ norm of $T_{zw}(s)$.

2 Main Results

2.1 Lossless Case ($C = 0$)

For a mass-spring system without damping, the optimal H_2 controller is another mass-spring system with damping $2I$:

$$M\ddot{q}_\mathrm{K} + 2\dot{q}_\mathrm{K} + Kq_\mathrm{K} = y, \quad q_\mathrm{K}(0) = \dot{q}_\mathrm{K}(0) = 0, \tag{3}$$

$$u = -\dot{q}_\mathrm{K}. \tag{4}$$

The optimal disturbance rejection gains are:

$$\gamma^*_{H_2} = \sqrt{2\mathrm{trace}(M^{-1})}, \quad \gamma^*_{H_\infty} = \sqrt{2}, \quad u = -\sqrt{2}y. \tag{5}$$

Notably, the H_2 optimal controller is a physical damped mass-spring system requiring no energy source, while the H_∞ controller is a simple decentralized proportional feedback [1].

2.2 Uniformly Damped Case

For systems with uniform damping, where C satisfies

$$C = M \sum_{j=0}^{n-1} \alpha_j (M^{-1}K)^j, \tag{6}$$

the optimal H_2 gain is

$$\gamma^*_{H_2} = \sqrt{\mathrm{trace}(Z_C^3) + \mathrm{trace}(Z_C)}, \tag{7}$$

where

$$Z_C = -\sqrt{M^{-1}}C\sqrt{M^{-1}} + \sqrt{\sqrt{M^{-1}}CM^{-1}C\sqrt{M^{-1}} + I}. \tag{8}$$

The optimal controller is another damped mass-spring system:

$$M_{\mathrm{K}}\ddot{q}_{\mathrm{K}} + C_{\mathrm{K}}\dot{q}_{\mathrm{K}} + K_{\mathrm{K}}q_{\mathrm{K}} = y, \quad u = -\dot{q}_{\mathrm{K}}, \tag{9}$$

where $M_{\mathrm{K}}, C_{\mathrm{K}}, K_{\mathrm{K}}$ are computed from M, C, and K [1].

3 Application to Power Systems

The results have direct applications in power systems modeled by the swing equation, which governs generator dynamics under lossless power transmission. The system behaves as a mass-spring network, where deviations from nominal frequency correspond to displacements in a mechanical analogy. The optimal H_∞ controller is simply a decentralized negative feedback proportional to frequency deviation, which aligns with standard droop control used in power networks. This result confirms the theoretical optimality of an already widely adopted control strategy.

References

[1] Johan Lindberg. "On H2 and H-infinity Optimal Control of Mass-Spring Networks with Power System Applications". English. Licentiate Thesis. Department of Automatic Control, Dec. 2023.