Data-Driven Estimation of Structured Singular Values

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I. Introduction

Robust control methods depend critically on the structured singular value μ, a measure of how large uncertainties can become before destabilizing a feedback system. Although μ is highly valuable, its exact computation is known to be NP-hard, motivating methods to estimate lower and upper bounds. Traditional approaches, such as MATLAB's Robust Control Toolbox, rely heavily on precise system models. Unfortunately, accurate modeling is often difficult or impractical, motivating purely data-driven alternatives. This work presents a novel purely data-driven algorithm for estimating lower bounds of the structured singular value directly from experimental data. Our method extends classical model-based power iteration methods by Doyle [1], enabling robustness analysis without explicit model identification. More details appear in [2].

II. Problem Setup

Consider a linear time-invariant square multivariable discrete-time dynamical system defined by its transfer function $\boldsymbol{G}(z)$. It is composed of a "nominal" stable and strictly proper model $\boldsymbol{G}_0(z)$ and a stable block $\boldsymbol{\Delta}(z)$ denoting a multiplicative uncertainty. Then,

$$\boldsymbol{Y}(z) = \underbrace{[\boldsymbol{I} + \boldsymbol{G}_0(z)\boldsymbol{\Delta}(z)]^{-1}\boldsymbol{G}_0(z)}_{=:\boldsymbol{G}(z)}\boldsymbol{U}(z).$$

Suppose that $\boldsymbol{\Delta}$ is known to be of the form $\mathrm{diag}(\delta_1\boldsymbol{I}_{r_1}, \dots, \delta_s\boldsymbol{I}_{r_s}, \boldsymbol{\Delta}_1, \dots, \boldsymbol{\Delta}_f)$, where $\delta_1, \dots, \delta_s \in \mathbb{C}$ and $\boldsymbol{\Delta}_1, \dots, \boldsymbol{\Delta}_f$ are stable dynamical systems of sizes $m_1 \times m_1$, ..., $m_f \times m_f$, respectively. s represents the number of *repeated scalar blocks* and f the number of *full blocks*. Then, the *structured singular value* of $\boldsymbol{G}_0$ is defined as

$$\mu_\Delta(\mathbf{G}_0) := \frac{1}{\min\left\{ \|\boldsymbol{\Delta}\|_\infty \colon \begin{array}{c} \boldsymbol{\Delta} \in \Delta, \\ \det\Big[\mathbf{I} + \mathbf{G}_0(z)\,\boldsymbol{\Delta}(z)\Big] = 0 \\ \text{for some } z \in \mathbb{T} \end{array} \right\}}.$$

Unfortunately, computing $\mu_\Delta(\boldsymbol{G}_0)$ is hard, hence one typically needs to rely on lower and upper bounds for it [1]. An appealing approach that exists in the literature [3], [4] is to compute a lower bound on $\mu_\Delta(\boldsymbol{G}_0)$ based on the power iterations method [5] and a model for $\boldsymbol{G}_0$. Inspired by that method, in this paper we propose instead a fully data-driven approach to compute a lower bound on $\mu_\Delta(\boldsymbol{G}_0)$ that does not require knowledge of $\boldsymbol{G}_0$.

III. Preliminaries

Let

$$\Delta := \{\mathrm{diag}(\delta_1\boldsymbol{I}_{r_1}, \dots, \delta_s\boldsymbol{I}_{r_s}, \boldsymbol{\Delta}_1, \dots, \boldsymbol{\Delta}_f) \colon \delta_1, \dots, \delta_s \in \mathbb{C},\ \boldsymbol{\Delta}_1 \in \mathbb{C}^{m_1 \times m_1}, \dots, \boldsymbol{\Delta}_f \in \mathbb{C}^{m_f \times m_f}\},$$
$$\mathrm{B}\Delta := \{\boldsymbol{\Delta} \in \Delta \colon \bar{\sigma}(\boldsymbol{\Delta}) \le 1\},$$

where $r_1, \dots, r_s, m_1, \dots, m_f$ are fixed positive integers such that $r_1 + \cdots + r_s + m_1 + \cdots + m_f = n$. Based on these sets, one can define the structured singular value at a specific frequency $\omega \in [-\pi, \pi)$ (at which we define $\boldsymbol{M} = \boldsymbol{G}_0(e^{i\omega})$):

Definition 1 (Structured singular value): For $\boldsymbol{M} \in \mathbb{C}^{n \times n}$, let

$$\mu'_\Delta(\boldsymbol{M}) := \frac{1}{\min\,\{\bar{\sigma}(\boldsymbol{\Delta}) \colon \boldsymbol{\Delta} \in \Delta,\ \det(\boldsymbol{I} + \boldsymbol{M}\boldsymbol{\Delta}) = 0\}}.$$

The following theorem establishes bounds on $\mu'_\Delta(\boldsymbol{M})$.

Theorem 1: For all $\boldsymbol{M} \in \mathbb{C}^{n \times n}$, we have that

$$\max_{\boldsymbol{Q} \in \mathcal{Q}} \rho(\boldsymbol{Q}\boldsymbol{M}) = \max_{\boldsymbol{\Delta} \in \mathrm{B}\Delta} \rho(\boldsymbol{\Delta}\boldsymbol{M}) = \mu'_\Delta(\boldsymbol{M}) \le \inf_{\boldsymbol{D} \in \mathcal{D}} \bar{\sigma}(\boldsymbol{D}\boldsymbol{M}\boldsymbol{D}^{-1}).$$

Motivated by Theorem 1, our goal is to derive a method for finding a local maximum of the function $\boldsymbol{\Delta} \mapsto \rho(\boldsymbol{\Delta}\boldsymbol{M})$ over all $\boldsymbol{\Delta} \in \mathrm{B}\Delta$.

IV. Proposed approach

We propose a novel data-driven approach to compute a lower bound on μ_Δ by adapting the power method given in [3]. Since we need to obtain a μ_Δ for each frequency, we carry out many of the operations in the frequency domain while the simulations of the plant are in time domain.

The pseudo-code for the data-driven estimation of Structured Singular Values is shown in Algorithm 1. The algorithm terminates when $\bar{\mu} \approx \tilde{\mu}$, and their values remain unchanged across iterations for each frequency m. Finally, $\mu_\Delta(G_0)$ is obtained by selecting the maximum μ across all frequencies.

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Algorithm 1 Data-Driven Estimation of Structured Singular Value

Require: Initial vectors $\boldsymbol{B}[0,:]$, $\boldsymbol{W}[0,:]$, Frequencies $m = 1, \dots, N$

for $l = 0, 1, \dots$ until convergence **do**
  Direct excitation:
  $\boldsymbol{b}[l,t] \leftarrow \text{ifft}(\boldsymbol{B}[l,m])$
  Excite $\boldsymbol{G}_0$ with $\boldsymbol{b}[l,t]$; measure $\boldsymbol{p}[l,t]$
  $\boldsymbol{P}[l,m] \leftarrow \text{fft}(\boldsymbol{p}[l,t])$
  $\tilde{\mu}[l+1,m] \leftarrow \|\boldsymbol{P}[l,m]\|$ for each frequency m
  Update $\boldsymbol{A}[l+1,m]$, $\boldsymbol{Z}[l+1,m]$ for each frequency m
  Transpose-plant excitation:
  $\boldsymbol{z}[l+1,t] \leftarrow \text{ifft}(\boldsymbol{Z}[l+1,m])$
  Set $\boldsymbol{r}[l+1,t] = 0$
  for $\alpha, \beta = 1, \dots, n$ **do**
    Excite $\boldsymbol{G}_0$ with $\boldsymbol{e}_\alpha \boldsymbol{e}_\beta^T \boldsymbol{z}[l+1,:]$; measure $\boldsymbol{g}[l+1,:]$
    $\boldsymbol{r}[l+1,:] \leftarrow \boldsymbol{r}[l+1,:] + \boldsymbol{e}_\alpha \boldsymbol{e}_\beta^T \boldsymbol{g}[l+1,:]$
  end for
  $\boldsymbol{R}[l+1,m] \leftarrow \text{fft}(\boldsymbol{r}[l+1,t])$
  $\bar{\mu}[l+1,m] \leftarrow \|\boldsymbol{R}[l+1,m]\|$ for each frequency m
  Update $\boldsymbol{W}[l+1,m]$, $\boldsymbol{B}[l+1,m]$ for each frequency m
end for
return $\mu_\Delta = \max_m \tilde{\mu}[m]$

Direct excitation: The computation of $\boldsymbol{P}$ can be carried out as follows:

$$\boldsymbol{b}[l,t] \leftarrow \frac{1}{N} \sum_{m=0}^{N-1} \boldsymbol{B}[l,m] e^{i2\pi mt/N},\ t = 1, \dots, N$$

$$\boldsymbol{P}[l,m] \leftarrow \sum_{t=1}^{N} \boldsymbol{G}_0(q) \boldsymbol{b}[l,t] e^{-i2\pi mt/N},\ m = 1, \dots, N.$$

Transpose-plant excitation: To account for the transpose of $\boldsymbol{G}_0$, we use the identity from [6, Eq. (21)]:

$$\boldsymbol{G}_0^T(q) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \boldsymbol{e}_\alpha \boldsymbol{e}_\beta^T \boldsymbol{G}_0(q) \boldsymbol{e}_\alpha \boldsymbol{e}_\beta^T.$$

Thus, we obtain the pseudo-code for computing $\boldsymbol{R}$:

for $\alpha,\ \beta = 1, \dots, n$:

$$\boldsymbol{r}[l+1,:] \leftarrow \boldsymbol{r}[l+1,:] + \boldsymbol{e}_\alpha \boldsymbol{e}_\beta^T \boldsymbol{G}_0(q) \boldsymbol{e}_\alpha \boldsymbol{e}_\beta^T \boldsymbol{z}[l+1,:]$$

$$\boldsymbol{R}[l+1,m] \leftarrow \sum_{t=1}^{N} \boldsymbol{r}[l+1,t] e^{i2\pi m(t-1)/N}, \quad m = 1, \dots, N.$$

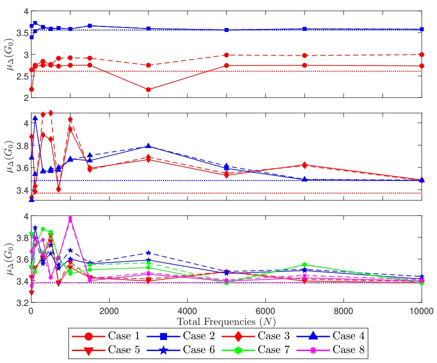

Fig. 1. Comparison between $\mu_\Delta(\boldsymbol{G}_0)$ and N. Solid line: $\tilde{\mu}$, dashed line: $\bar{\mu}$, dotted line: `mussv`. From top to bottom, systems with 1 block, 2 blocks, and 3 blocks, respectively.

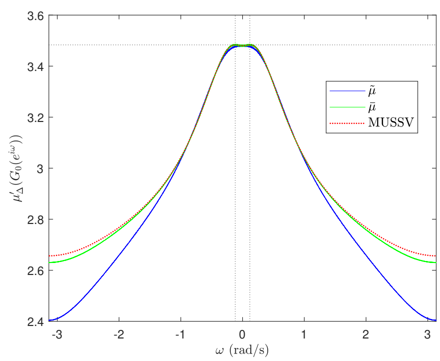

Fig. 2. Structured singular value for different frequencies.

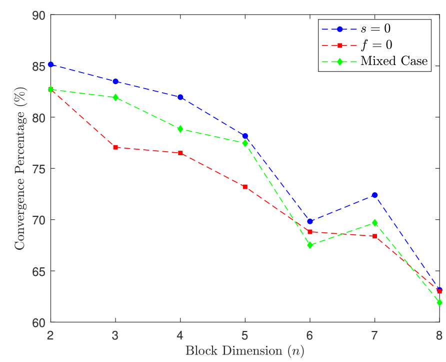

Fig. 3. Accuracy improvement with frequency samples.

V. Experiments and Conclusion

Figure 1 displays the results of a heuristic example with $\boldsymbol{\Delta} \subset \mathbb{C}^{3\times 3}$, where eight different block structures are analyzed in terms of convergence to the lower bound μ_M, provided by the `mussv` command in MATLAB, and its dependence on the number of frequency samples N. Extensive simulations show that the algorithm generally exhibits good convergence when $f > s$, $m_k > r_j$ (for all j, k), and $s = 0$ with $f \geq 1$. Note that, as N increases, the performance of Algorithm 1 improves. Figure 2 illustrates the structured singular value for a single repeated scalar block of dimension $r_1 = 1$ and a single full block with $m_1 = 2$ (Case 4 in Figure 1), where $\tilde{\mu}$ and $\bar{\mu}$ align well with μ_M at the dominant frequency but deviate at others, which is expected based on the properties of the power method [7].

Algorithm 1 has also been tested on a large set of randomly generated matrices. A total of 2100 experiments have been conducted for three block structure configurations ($s = 0$, $f = 0$ and a mixed case) on $n \times n$ complex matrices, with $n = 2, 3, \dots, 8$. Figure 3 illustrates the percentage of simulations where the average of $\tilde{\mu}$ and $\bar{\mu}$ converge to μ_M, which demonstrates better performance for $s = 0$ and, across all three cases, a deterioration of the algorithm when $n > 5$.

Numerical examples show our method closely matches `mussv` lower bounds. Future studies could extend evaluation to real-world systems to further assess robustness.

References

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