Symmetrizable systems

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I. Introduction

Transforming an asymmetric system into a symmetric system makes it possible to exploit the simplifying properties of symmetry in control problems. We define and characterize the family of symmetrizable systems, which can be transformed into symmetric systems by a linear transformation of their inputs and outputs. In the special case of complete symmetry, the set of symmetrizable systems is convex and verifiable by a semidefinite program. A Khatri-Rao rank needs to be satisfied for a system to be symmetrizable. Therefore, linear systems are generically neither symmetric nor symmetrizable.

II. Symmetric systems

Symmetric systems, also known as reciprocal systems in the literature, are control systems that exhibit input-output symmetry in their transfer function matrices $G(s) \in \mathbb{C}^{m\times m}$ as follows

$$\Sigma_e G^\top(s) = G(s)\Sigma_e, \tag{1}$$

where Σ_e is a constant diagonal matrix whose diagonal elements are either 1 or -1. Relaxation systems are a well-known subset of these systems with $\Sigma_e = I$.

Symmetric systems are found in diverse application areas, such as electrical circuits, chemical reactors, mechanical systems, and power networks. These systems have properties that simplify many control problems. For example, certain H_2 and H_∞ optimal control problems have well-known analytical solutions when applied to symmetric systems, the optimal linear-quadratic regulator of symmetric systems can be obtained by iterative learning control with no bias, and estimated by a single trajectory of the system. To be able to use these simplifying properties beyond symmetric systems, we need to extend the family of symmetric systems.

III. Symmetrizable systems

We extend the family of symmetric systems, by introducing symmetrizable systems. These systems may not be symmetric, but after a linear transformation of their inputs and outputs as

$$H(s) = K^{-1}G(s)K, \tag{2}$$

they become symmetric, where $K \in \mathbb{R}^{m\times m}$ is a constant invertible matrix, called the symmetrizing gain. The transformation (2) induces the following equivalence relation on $m \times m$ transfer functions: $G \overset{\mathrm{s}}{\sim} H$ is true if there is some non-singular $K \in \mathbb{R}^{m\times m}$ such that (2) holds. Hence, if a member of a class is symmetric, all members of that class are symmetrizable. If no member of a class is symmetric, then none of the members are symmetrizable. The following theorem provides the conditions that characterize all symmetrizable systems.

Theorem 1: A system represented by the state space

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is symmetrizable if and only if the following equations have a non-singular solution for Q:

$$PQ = QP^\top,\ Q = Q^\top \tag{3}$$

$$Q_{12} = 0. \tag{4}$$

where $Q_{12} = [I_n\ \ 0]\, Q\, [0\ \ I_m]^\top$ is an off-diagonal block of Q.

Remark 1: Symmetrizable systems include symmetric systems as a subset. This follows from choosing $K = I$ in (2).

Remark 2: When the system represented by P is symmetrizable, one can find a symmetrizing gain K and a symmetric realization T based on the solution Q of the equations (3)-(4). Such matrices T and K are not unique, with one instance being

$$T = F_1|D_1|^{1/2},\ K = F_2|D_2|^{1/2} \tag{5}$$

where D_1 (D_2) is the diagonal matrix of eigenvalues of Q_{11} (Q_{22}) associated with an orthonormal matrix of eigenvectors F_1 (F_2) sorted such that $\mathrm{sgn}(D_1) = -\Sigma_i$ ($\mathrm{sgn}(D_2) = \Sigma_e$).

Remark 3: Enforcing $Q \succ 0$ in (3) makes the symmetrizability conditions convex. A system can be symmetrized into a completely symmetric system if and only if the following semidefinite program is feasible

$$\begin{array}{rl} \text{find} & Q \in \mathbb{R}^{(n+m)\times(n+m)} \\ \text{subject to} & Q \succ 0 \\ & PQ = QP^\top \\ & [I_n\ \ 0]\, Q\, [0\ \ I_m]^\top = 0. \end{array} \tag{6}$$

IV. Examples

A. A classical multi-tank system

In this example, we show how symmetrizability extends symmetry in a physical system. Consider a quadruple-tank process with two inputs (voltages applied to the pumps) and two outputs (levels of the lower tanks). This process can be described by the transfer function

$$G(s) = \begin{bmatrix} \frac{c_{11}}{1+sT_1} & \frac{c_{12}}{(1+sT_1)(1+sT_3)} \\ \frac{c_{21}}{(1+sT_2)(1+sT_4)} & \frac{c_{22}}{1+sT_2} \end{bmatrix}, \quad (7)$$

where

$$\begin{aligned} c_{11} &= \gamma_1 k_1 T_1 k_c / A_1, \\ c_{12} &= (1-\gamma_2) k_2 T_1 k_c / A_1 \neq 0, \\ c_{21} &= (1-\gamma_1) k_1 T_2 k_c / A_2 \neq 0, \\ c_{22} &= \gamma_2 k_2 T_2 k_c / A_2. \end{aligned} \quad (8)$$

In (7)-(8), A_i is the cross-section of the ith tank, T_i is the time constant associated with the ith tank, constants $k_i > 0$ and $\gamma_i \in [0,1]$ are determined by the ith pump and valve settings and $k_c > 0$. From definition (1), this system is symmetric if and only if

$$T_1 = T_2, T_3 = T_4 \text{ or } T_1 = T_4, T_2 = T_3 \quad (9)$$

$$c_{12} = \pm c_{21}. \quad (10)$$

However, the first condition (9) is sufficient for the system to be symmetrizable. Hence symmetrizability imposes a less restrictive condition on the tank parameters than symmetry.

B. Applications in optimal control

In this example, we show how the simplifying properties of symmetric systems carry over to symmetrizable systems. This is realized by first transforming the asymmetric system $G(s)$ to a symmetric system $H(s)$ as (2), solving a control problem for $H(s)$, and converting the result back for $G(s)$.

For example, consider the static output-feedback controller $u_s(t) = K_s\, y_s(t)$ designed for the symmetric system $H(s)$ with input u_s and output y_s. This control law is equivalent to applying the following static output feedback to $G(s)$:

$$u(t) = K K_s K^{-1} y(t), \quad (11)$$

where u and y are the input and output of the asymmetric system $G(s)$, respectively. To demonstrate this application, we consider the linear system described by the minimal state-space equations

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bu(t) + w(t) \\ y(t) &=& Cx(t) + Du(t) \end{array}, \quad t \geq 0 \quad (12)$$

where $x(0) = 0$ and $w(t) \in \mathbb{R}^n$ is the disturbance input. We would like to find an output-feedback linear controller that stabilizes the system and minimizes the performance measure

$$\mathcal{J}(R,S) = \sup_{w \in \mathcal{W}(S)} \int_0^\infty y(t)^\top R y(t) + \alpha^2 u(t)^\top R u(t) dt$$

where $\alpha > 0$ balances the control effort versus output regulation in the objective function and

$$\mathcal{W}(S) = \left\{ w \,\middle|\, \int_0^\infty w^\top(t) S w(t) dt \leq 1 \right\},$$

where $R, S \succ 0$. This problem was introduced and solved for relaxation systems in [1], for which $R = I$. We extend this result to a family of *symmetrizable* systems that include relaxation systems as a proper subset and allow for $R \succ 0$ as any positive definite matrix.

Now, assume that system (12) is symmetrizable with complete symmetry. Then the semidefinite program (6) has a solution $Q \succ 0$ which can be used find

$$T = Q_{11}^{1/2},\ K = Q_{22}^{1/2}.$$

Choosing

$$R = K^{-2},\ S = T^{-2}$$

and following the results of [1], shows that the performance measure $\mathcal{J}(K^{-2}, T^{-2})$ is minimized with a static output-feedback controller with the closed-form expression

$$u_s(t) = -\alpha^{-1}(K^{-1}DK - K^{-1}CA^{-1}BK) y_s(t).$$

Therefore, the optimal feedback controller from y to u is given by

$$u(t) = -\alpha^{-1}(D - CA^{-1}B) y(t).$$

V. Conclusion

We studied the problem of transforming a linear system into a symmetric system using a static gain. We have derived the conditions for this transformation to exist and provided a method to obtain both the symmetrizing gains and symmetric realizations with different signatures.

Symmetrizability is a weaker condition than symmetry. Yet, it allows one to apply the simplifying properties of symmetry when controlling symmetrizable systems. To demonstrate this point, we revisited an optimal control problem and extended its analytic solution for symmetric systems to symmetrizable systems.

References

[1] Richard Pates, Carolina Bergeling, and Anders Rantzer. "On the optimal control of relaxation systems". In: *58th conference on decision and control (CDC)*. IEEE. 2019, pp. 6068–6073.