Leader selection and control design for topology estimation of dynamical networks

Nana Wang and Dimos V. Dimarogonas

I. Introduction

Networks of dynamical systems represent many physical systems, including power systems, biological systems, sensor systems, transport systems, etc. [1]. The topological network structures, displaying important properties of the networks and playing a crucial role in understanding the behaviors of the networks [2], are the basis for further performance improvement. In many practical systems, there is a lack of detailed topological structures, and hence additional efforts are needed to estimate them. For example, in the human immune system, inference of communication between different types of cells remains a challenging problem.

Main Contributions: We propose a systematic method for estimating unknown edge weights by controlling a possible leader set, integrating leader selection and controller design to effectively excite the network. The designed control scheme only excites the selected leader nodes, and the proposed parameter estimation scheme ensures that edge weight estimation errors become zero in finite time.

II. Problem formulation

The dynamics of a network is represented as

$$\dot{x} = Ax + Bu \tag{1}$$

where $x = [x_1, x_2, \cdots, x_N]^T \in R^N$ are the states of the network; N is the number of agents; A could be a Laplacian matrix of the directed weighted graph $\mathcal{G}$ in [1]; A could also be a general topological matrix here; For $A_{ij} = 0$, it means that there is no connection among v_i and v_j, where $i, j = 1, 2, \cdots, N, i \neq j$; For $A_{ij} \neq 0$, there is an edge among node v_i and v_j, where $i, j = 1, 2, \cdots, N, i \neq j$; A_{ii} denotes that its own effect for each $i = 1, 2, \cdots, N$; The diagonal matrix $B \in R^{N\times N}$ with $B_{ii} \in \{0, 1\}$ for $i = 1, \cdots, N$, is a design parameter; $u \in R^N$ denotes the external control input. Here we call the node i as the leader if $B_{ii} \neq 0$.

The objective is to choose B and design the external control u on these leader nodes which stabilizes and excites the network to estimate the unknown A.

This work is supported by the Swedish Research Council (VR), the Knut and Alice Wallenberg Foundation, and the WASP-DDLS program.
N. Wang and D. V. Dimarogonas are with the Division of Decision and Control Systems, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden, email: {nanaw,dimos}@kth.se.

III. A novel algorithm for uniquely determining the topology

This part, we will design a new algorithm to find the unique solution of the topology matrix by building an auxiliary network based on leader selection method detailed in [3].

Choose $A_m \in R^{N\times N}$ as a reference matrix for A, where there is no requirement for A_m. Similar to the original network (1), we establish an auxiliary network as

$$\dot{\hat{x}} = A_m x + B^* u + \tau\tilde{x}, \tag{2}$$

with $\hat{x}(0) = 0$ and τ a positive constant, and the prediction error is $\tilde{x} = x - \hat{x}$.

A filter $w \in R^N$ for x is designed as

$$\dot{w} = x - \tau w, \tag{3}$$

with $w(0) = 0$.

Denote the error between the reference matrix A_m and A as $\tilde{A} = A_m - A$. Based on the original network (1) and the auxiliary network (2), the prediction error $\tilde{x}$ is

$$\dot{\tilde{x}} = -\tilde{A}x - \tau\tilde{x}, \tag{4}$$

with $\tilde{x}(0) = x(0)$. Denote $\zeta := \tilde{x} + \tilde{A}w$ and its time derivative yields

$$\dot{\zeta} = \dot{\tilde{x}} + \tilde{A}\dot{w} \tag{5}$$

From the derivative of the prediction error $\tilde{x}$ detailed in (4) and the derivative of w in (3), the time derivative $\dot{\zeta}$ becomes

$$\dot{\zeta} = -\tau(\tilde{x} + \tilde{A}w) = -\tau\zeta \tag{6}$$

with $\zeta(0) = \tilde{x}(0)$. The second equation is derived from $\zeta = \tilde{x} - \tilde{A}w$.

Construct two matrix functions $Y(t)$ and $Z(t)$ as

$$\begin{aligned} \dot{Y} &= ww^\top, \\ \dot{Z} &= (A_m w + \tilde{x} - \zeta)w^\top \end{aligned} \tag{7}$$

with $Y(0) = 0$ and $Z(0) = 0$.

With $Y(0) = 0$, integrate the derivative $\dot{Y}$ from 0 to t to obtain

$$Y(t) = \int_0^{t_e} w(\tau)w(\tau)^\top d\tau. \tag{8}$$

Define the topology estimation time t_e

$$t_e := \inf\{t > 0 | Y(t) \succ \gamma I\}. \tag{9}$$

with $\gamma > 0$.

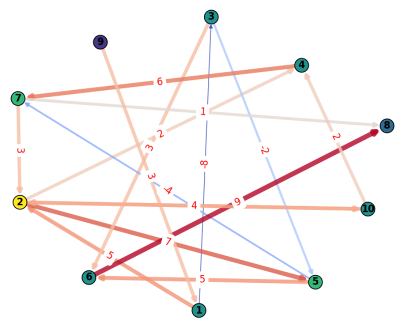

Fig. 1: Directed weighted graph

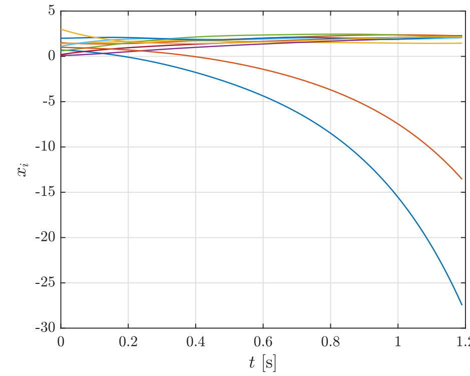

Fig. 2: The evolution of state x_i

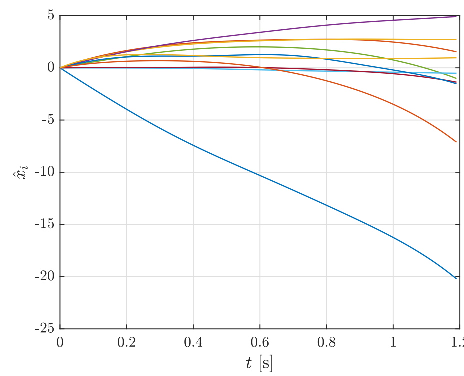

Fig. 3: The evolution of $\hat{x}_i$ in the auxiliary network

The weight estimation $\hat{A}(t)$ is then estimated as

$$\hat{A}(t) = Z(t)Y^{-1}(t), \tag{10}$$

for all $t \geq t_e$.

Condition 1: Design a controller $u(t) = (u_1(t), \ldots, u_{M_1}(t))^\top \in R^{M_1}$ where M_1 is the number of the selected leader nodes: $u_i(t)$ is designed to be bounded, stationary and sufficiently rich of order n for $i = 1, \ldots, M_1$.

Now we can provide the result of accurately estimating the topology matrix A after t_e.

Theorem 1: Assume an asymptotically stable A for the network (1) and (A, B) is controllable. Design a controller u that satisfies Condition 1. Then, if we implement the weight estimation algorithm (10), the weight estimation error $\tilde{A}' = \hat{A} - A$ will be zero after t_e. □

Theorem 2: Assume that (A, B) is controllable and choose the input u thar is bounded, $u^{(n-r)}$ is piecewise uniformly continuous with minimum interdiscontinuity distance $\kappa > 0$ and PE of order $n - r + 1$. Then implementing the weight estimation algorithm (10) guarantees that the weight estimation error $\tilde{A}' = \hat{A} - A$ will be zero after t_e, where t_e is the same as in (9). □

IV. Simulation

For the simulation example, we considered 10 agents whose communication network is represented by a weighted directed graph depicted in Fig. 1. We used the Laplacian matrix of this weighted graph $\mathcal{G}$ for A and $\bar{A}$ is the Laplacian matrix of the corresponding directed graph $\mathcal{D}$. The number of the edges in Fig. 1 denotes their edges' weights. The input matrix B^* is determined as

$$B^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The parameter of the (3) is $\tau = 4$. The control input $u(t)$ is $\begin{bmatrix} \cos(\pi/2t - \pi/8) - \sin(\pi t + 2\pi/3) + \cos(2\pi t) + \sin(\pi t) \\ \sin(2\pi t) - \sin(3\pi t - \pi/3) + \cos(4\pi t - \pi/4) - \cos(\pi t) \end{bmatrix}$.

Set A_m of the auxiliary network (2) as

$$\begin{bmatrix} -2 & -3 & -3 & -4 & -5 & -3 & -4 & -3 & & \\ & -2 & 1 & & & 2 & & & 1 & 1 \\ & & -1 & -1 & 1 & 1 & & 1 & & 1 \\ & & & -1 & 1 & 1 & 2 & 2 & & 1 \\ & & & & -1 & & 1 & 1 & 2 & \\ & & & & & -2 & -2 & -1 & & \\ & & & & & & -2 & -2 & 1 & 1 \\ 1 & 2 & & & & & & -2 & & \\ & 1 & 2 & & & -2 & & & & \\ & -2 & & -2 & & & & 1 & & 1 \end{bmatrix}$$

We set the initial value as $x(0) = [0.7; 1.5; 1.4; 0.1; 0.6; 1.1; 0.2; 2; 1; 3]$. The Laplacian matrix A has a positive eigenvalue 3, which makes the network (1) unstable. We obtained the topology estimation time $t_e = 1.12s$ when setting $\gamma = 1e - 12$ in (**??**) and $\|\tilde{A}\| = 8.06e - 05$, which verifies Theorem 2. When setting different γ, we get different t_e. The smaller γ we set, the smaller t_e it is. The trajectory of state x is displayed in Fig.2 and the evolution of $\hat{x}$ of the auxiliary network is shown in Fig.3 during $t \in [0, 1.2]s$. From them, one can witness that the state x and $\hat{x}$ of the auxiliary network are still bounded before the topology estimation is achieved.

References

[1] M. Mesbahi and M. Egerstedt, *Graph theoretic methods in multiagent networks*. Princeton University Press, 2010.

[2] E. Tegling, R. H. Middleton, and M. M. Seron, "Scalability and fragility in bounded-degree consensus networks," *IFAC-PapersOnLine*, vol. 52, no. 20, pp. 85–90, 2019.

[3] W. Abbas, M. Shabbir, Y. Yazıcıoğlu, and X. Koutsoukos, "Leader selection for strong structural controllability in networks using zero forcing sets," in *2022 American Control Conference (ACC)*. IEEE, 2022, pp. 1444–1449.