Wisdom of crowds in signed DeGroot models

Muhammad Ahsan Razaq[1] and Claudio Altafini[1]

I. Introduction

The "wisdom of crowds" effect refers to the phenomenon in which the collective judgment of a group can become more accurate and reliable than that of the individual agents acting in isolation [1]. In the DeGroot model, individuals update their opinions by averaging those of their neighbors. The opinions reach a consensus which is a weighted average of the initial opinions. The weights are called the social power of the individuals. Tian *et al.* [2] investigate how changes in the social power vector influence wisdom and provide a description of the convex region inside which the social power vector improves the wisdom of the crowd. The paper [2] focuses on positive interactions between individuals. Recently, DeGroot model has been extended to antagonistic networks [3], [4]. Here, we investigate the wisdom of crowds on signed DeGroot models [4]. For the DeGroot model, we analyze in detail two distinct cases: *signed unipartite case* and *signed bipartite case*. We show that the region of improvement increases for the signed unipartite case but that we converge to the false truth with great confidence for the signed bipartite case.

II. Problem Formulation

Consider n individuals engaged in a discussion. The DeGroot model [5] models the discussion:

$$\mathbf{x}(k+1) = W\mathbf{x}(k). \tag{1}$$

where $\mathbf{x}(k)$ is the opinion vector at time k and W is the interaction pattern with $W \geq 0$, $W\mathbb{1} = \mathbb{1}$ and $1 \in \Lambda(W)$ a simple and strictly dominant eigenvalue. Initially, individual i opinion, $x_i(0)$, has mean ζ (the true value) and bounded variance $\mathrm{Var}[x_i(0)] = \sigma_i^2$.

The collective wisdom at $t = 0$ can be quantified by the group mean, $\mathbb{E}[\overline{\mathbf{x}}(0)] = \zeta$ and the group variance $\mathrm{Var}[\overline{\mathbf{x}}(0)] = \frac{1}{n^2}\sum_{i=1}^n \sigma_i^2$ where $\overline{\mathbf{x}}(0) = \frac{1}{n}\sum_{i=1}^n x_i(0)$.

For a standard DeGroot model, the opinions converge to consensus

$$\mathbf{x}^* = \mathbf{y}^T\mathbf{x}(0)\mathbb{1}, \tag{2}$$

where the consensus value is influenced by the social power vector $\mathbf{y}$, the dominant left eigenvector of W. Since $W \geq 0$, the social power vector is $\mathbf{y} > 0$ with $\mathbf{y}^T\mathbb{1} = 1$.

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[1] The authors are with the Division of Automatic Control, Department of Electrical Engineering, Linköping University, SE-58183 Linköping, Sweden (email: muhammad.ahsan.razaq@liu.se; claudio.altafini@liu.se)

As individuals update their opinions, the collective wisdom changes. At steady state, the group mean is $\mathbb{E}[\overline{\mathbf{x}}^*]$ and the group variance is $\mathrm{Var}[\overline{\mathbf{x}}^*] = \sum_{i=1}^n y_i^2\sigma_i^2$ where $\overline{\mathbf{x}}^* = \mathbf{y}^T\mathbf{x}(0)$.

To quantify the wisdom of the crowd of the DeGroot model, we say that the DeGroot model is *Mean accurate* if $\mathbb{E}[\overline{\mathbf{x}}^*] = \zeta$; *Concentrating* if $\mathrm{Var}[\overline{\mathbf{x}}^*] < \mathrm{Var}[\overline{\mathbf{x}}(0)]$; and *Dispersing* if $\mathrm{Var}[\overline{\mathbf{x}}^*] > \mathrm{Var}[\overline{\mathbf{x}}(0)]$.

A mean accurate and concentrating model is said to "improve the wisdom of crowd", while a mean accurate and dispersing model is said to "undermine the wisdom of crowd" [2].

The model (1) is mean accurate as $\mathbb{E}[\mathbf{x}^*] = \zeta$. It is concentrating if

$$n^2\sum_{i=1}^n y_i^2\sigma_i^2 < \sum_{i=1}^n \sigma_i^2 \tag{3}$$

and dispersing if

$$n^2\sum_{i=1}^n y_i^2\sigma_i^2 > \sum_{i=1}^n \sigma_i^2. \tag{4}$$

The concentrating condition (Eq. (3)) leads to a region inside the hyperellipse

$$\Phi_1 = \left\{ \frac{n^2\sum_{i=1}^n y_i^2\sigma_i^2}{\sum_{i=1}^n \sigma_i^2} < 1 \right\}.$$

Since $\mathbf{y}^T\mathbb{1} = 1$ and the social power vector lies in the $n-$simplex ($\mathbf{y} \in \Delta$, where $\Delta = \{\mathbf{y} | \mathbf{y}^T\mathbb{1} = 1, \, \mathbf{y} > 0\}$), the model (1) improves the wisdom of the crowd if $\mathbf{y} \in \Gamma_1$ and undermines the wisdom of the crowd if $\mathbf{y} \notin \Gamma_1$ where $\Gamma_1 = \Delta \cap \Phi_1$.

The region Γ_1 depends on the variances σ_i^2. For instance, the green region in Figure 1 shows the convex region Γ_1 for $\sigma_i^2 = \{6, 1, 1\}$.

III. Wisdom of Crowds in Signed Unipartite Networks

In this section, we consider the interaction matrix W which may have negative terms, i.e. $W \gtreqless 0$. Unipartite refers to the fact that the model (1) still converges to the consensus value (2), but now $\mathbf{y} \gtreqless 0$ with $\mathbf{y}^T\mathbb{1} = 1$.

In the unipartite case, the model (1) remains mean accurate, i.e., $\mathbb{E}[\overline{\mathbf{x}}^*] = \zeta$ with group variance $\mathrm{Var}[\overline{\mathbf{x}}^*] = \sum_{i=1}^n y_i^2\sigma_i^2$. Here, $\mathbf{y} \gtreqless 0$ implies that the social power vector lies in the hyperplane $\Psi_1 = \{\mathbf{y} | \mathbf{y}^{\mathbf{T}}\mathbb{1} = \mathbb{1}\}$ instead of the $n-$simplex Δ.

The wisdom of the crowd improves if $\mathbf{y} \in \Gamma_2$ and undermines if $\mathbf{y} \notin \Gamma_2$ where $\Gamma_2 = \Psi_1 \cap \Phi_1$.

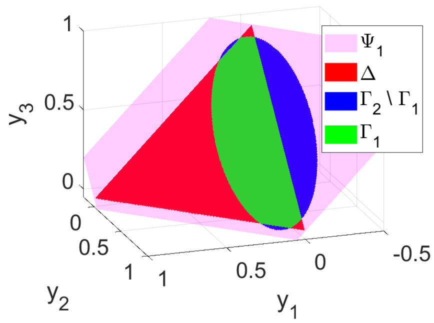

Fig. 1: Concentration regions Γ_1 and Γ_2 corresponding to an hyperellipsoid determined by the variances $\sigma_i^2 = \{6, 1, 1\}$. The hyperplane Ψ_1 is shown in pink, the 3-simplex Δ is shown in red, the region Γ_1 is in green while Γ_2 includes both the regions in blue and in green.

We observe that $\Gamma_1 \subseteq \Gamma_2$, meaning that the region for potential improvement in collective wisdom is larger (or equal) in the signed unipartite case compared to the unsigned case. For instance, Figure 1 shows the convex region Γ_2 (in blue and green) for $\sigma_i^2 = \{6, 1, 1\}$, extending beyond the 3−simplex.

Example 1 Consider an interaction matrix

$$W = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ -0.5 & 0.9 & 0.6 \\ 0.9 & 0.4 & -0.3 \end{bmatrix}.$$

The social power vector for the DeGroot model is $\mathbf{y}^T = [-0.117,\ 0.7766,\ 0.3404]$. For initial variances $\sigma_i^2 = \{6, 1, 1\}$, the variance of the average of final opinions $\mathrm{Var}[\overline{\mathbf{x}}^*] = 0.8 < \mathrm{Var}[\overline{\mathbf{x}}(0)] = 0.89$, thus improving the wisdom of the crowd. ■

IV. Wisdom of Crowd in Signed Bipartite Networks

In this section, we consider a matrix W with the strictly dominant right eigenvector $\mathbf{v}$ which has some negative entries, i.e., $W\mathbf{v} = \mathbf{v}$ of components $v_i = \pm 1\ \forall i$. In this case, the model (1) converges to bipartite consensus

$$\mathbf{x}^* = \mathbf{y}^T \mathbf{x}(0)\mathbf{v},$$

with the social power vector $\mathbf{y} \gtreqless 0$ and $\mathbf{y}^T\mathbf{v} = 1$. The model (1) is not mean accurate unless $\mathbf{y}^T\mathbb{1} = n(\mathbb{1}^T\mathbf{v})^{-1}$ as

$$\mathbb{E}[\overline{\mathbf{x}}^*] = \mathbb{1}^T\mathbf{v}\mathbf{y}^T\mathbb{1}\zeta/n$$

with $\mathbf{y}^T\mathbb{1} \neq 1$ and $|\mathbb{1}^T\mathbf{v}/n| < 1$. The group variance is

$$\mathrm{Var}[\overline{\mathbf{x}}^*] = (\mathbb{1}^T\mathbf{v}/n)^2 \sum_{i=1}^{n} y_i^2\sigma_i^2.$$

So, the opinions almost surely concentrate around the false truth $\mathbb{1}^T\mathbf{v}\mathbf{y}^T\mathbb{1}\zeta/n$ if $\mathbf{y} \in \Gamma_3$ and disperse if $\mathbf{y} \notin \Gamma_3$ where $\Gamma_3 = \Psi_2 \cap \Phi_2$, $\Psi_2 = \{\mathbf{y} \in \mathbb{R}^n | \mathbf{y}^T\mathbf{v} = 1\}$ and

$$\Phi_2 = \left\{ \frac{\sum_{i=1}^n y_i^2\sigma_i^2}{\sum_{i=1}^n \sigma_i^2} < (\mathbb{1}^T\mathbf{v})^{-2} \right\}.$$

As the sizes of the opinion bipartition approach each other, $\mathbb{1}^T\mathbf{v}$ decreases, and the size of the hyperellipsoid Φ_2 increases. This leads to a greater concentration of variance and consequently enlarges the convex region Γ_3.

This means that the individuals become more convinced of the false truth compared to the signed unipartite case, where opinions instead converge around the true value.

Example 2 Consider an interaction matrix

$$W = \begin{bmatrix} 0.3 & -0.6 & -0.1 \\ -0.3 & 0.8 & -0.1 \\ -0.2 & 0.9 & -0.1 \end{bmatrix}.$$

The social power vector is $\mathbf{y}^T = [0.3039,\ -0.7353,\ 0.0392]$ with the dominant right eigenvector $\mathbf{v}^T = [1,\ -1,\ -1]$. If the initial opinions $\mathbf{x}(0)$ have mean $\zeta = 5$ and variances $\sigma_i^2 = \{4, 1, 8\}$, the variance of the average opinion at the start of the discussion is $\mathrm{Var}[\overline{\mathbf{x}}(0)] = 1.44$. The average of the individuals concentrates around the false truth $\mathbb{E}[\overline{\mathbf{x}}^*] = 0.6536$, with variance $\mathrm{Var}[\overline{\mathbf{x}}^*] = 0.1025$. If instead we consider the unipartite network, i.e.,

$$W = \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.3 & 0.8 & -0.1 \\ 0.2 & 0.9 & -0.1 \end{bmatrix}$$

with $\mathbf{y}^T = [0.3039,\ 0.7353,\ -0.0392]$, then the opinions concentrate around the true value $\mathbb{E}[\overline{\mathbf{x}}^*] = 5$ with the variance $\mathrm{Var}[\overline{\mathbf{x}}^*] = 0.9224$. ■

V. Conclusion

In this extended abstract, we analyze the conditions under which the wisdom of crowds gets improved or undermined by a DeGroot model with antagonistic interactions. We show that the region of improvement is always larger than in the unsigned case because improvement can occur also for negative social powers. Moreover, we show that in a bipartite setting the crowd typically converges to a false truth, around which the variance tends to concentrate faster than in the unipartite case.

References

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