Linear Regulator-based synchronization of positive discrete-time multi-agent systems ⋆

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Abstract: This extended abstract addresses the positive synchronization of interconnected systems on undirected graphs. For homogeneous positive systems, a static feedback protocol design is proposed, based on the Linear Regulator problem. The solution to the algebraic equation associated to the stabilizing policy can be found using a linear program. Simulations on large regular graphs with different nodal degree illustrate the proposed results.

Keywords: Multi-agent Systems, Optimal Control, Positive Systems, Dynamical consensus.

1. INTRODUCTION

1.1 Motivation

Synchronization is a critical behavior in many dynamical systems and has broad applications across various domains Ren and Beard (2005); Tegling et al. (2023); Fabiny et al. (1993); Fax and Murray (2004). This work investigates discrete-time multi-agent systems with homogeneous, linear time-invariant dynamics, focusing on achieving synchronization of agent states using only relative measurements.

The synchronization approach relies on solving the discrete-time Linear Regulator problem Gurpegui et al. (2024),which is analogous to the algebraic Riccati equation in the Linear Quadratic Regulator framework for positive systems Berman and Plemmons (SIAM, 1994); Luenberger (1979). The primary contributions of this work include a static feedback protocol derived from the Linear Regulator problem, which is solvable via linear programming (Protocol 1), and necessary and sufficient conditions ensuring the positivity of each agent's trajectory for any nonnegative initial conditions.

2. MODEL DESCRIPTION

2.1 Graph Description

This work focuses on families $\mathcal{F} \subset \mathbb{G}$ of connected, undirected graphs. Among these graph families, we are specifically interested in those for which the eigenvalues of the associated regularized Laplacian matrix $\mathcal{D}$ (Saberi et al., 2022, Ch. 3) satisfy upper and lower bounds.

Definition 1. The set of undirected graphs for which the associated row-stochastic matrix $\mathcal{D}$ satisfies the property that all eigenvalues, except for $\mu_1 = 1$, have an absolute value smaller than β and greater than γ is defined by

$$\mathbb{G}_{[\gamma,\beta]} = \{\mathcal{G} \in \mathbb{G} \,|\, \gamma \leq \mu_i(\mathcal{G}) \leq \beta, \ \forall i > 1\}.$$

where $\gamma \in (-1, \beta]$, $\beta \in (0, 1)$.

Expressions for γ and β has been studied in the literature—See *e.g.* Banerjee and Mehatari (2016).

2.2 Multi-agent Systems

We consider homogeneous multi-agent system (MAS) composed by an arbitrary number of identical, linear time-invariant positive agents. The dynamics of each agent $i = 1, \ldots, N$ are described by

$$x_i(t+1) = Ax_i(t) + Bu_i(t); \quad A \in \mathbb{R}_+^{n\times n}, B \in \mathbb{R}^{n\times m} \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are, respectively, the state and input vectors of agent i.

Agents access the relative information of their neighbors through full-state measurements. Specifically, each agent i has access to the quantity

$$\zeta_i(k) = \frac{1}{1+\sum_{j=1}^N w_{ij}} \sum_{k\in\mathcal{N}_i} w_{ik}(y_i(t) - y_k(t)). \quad (2)$$

3. PROBLEM FORMULATION AND MAIN RESULT

This paper addresses the following two problems:

Problem 1. (Synchronization Problem). Design a linear feedback controller of the form

$$u_i(t) = F\zeta_i(t) \quad (3)$$

that achieves state synchronization, i.e.

$$\lim_{t\to\infty} [x_i(t) - x_j(t)] = 0, \quad \forall i, j \in \{1, \ldots, N\}. \quad (4)$$

while satisfying the constraint:

$$|u_i| \leq E\,|\zeta_i|, \quad (5)$$

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where E is a design matrix that enforces bounded control inputs based on the relative measurements ζ_i.

Protocol 1. (LR-based protocol). Consider the MAS described by (1) and (2) with $A \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $E \in \mathbb{R}_+^{m\times n}$, $s \in \mathbb{R}_+^n$ such that $s > 0$ and $\mathcal{D}$ be the row stochastic matrix associated with a graph $\mathcal{G} \in \mathbb{G}_{[\gamma,\beta]}$ with N agents. Suppose

$$A - (1-\gamma)|B|E \geq 0 \tag{6}$$

$\gamma \in (-1, 1)$. The LR-based protocol is given by

$$u_i = -\rho K \zeta_i, \tag{7}$$

where $\rho \geq \frac{1}{1-\beta}$, $\beta \in (0, 1)$ and K follows from the Linear Regulator setting in Li and Rantzer (2024) with $\tilde{A} = A$, $\tilde{B} = B$, $\tilde{E} = \frac{1}{\rho}E$ and $s > 0$.

Theorem 2. Consider a family of graphs $\mathcal{F} \subseteq \mathbb{G}_{[\gamma,\beta]}$ and the MAS described by (1) and (2). If the pair (A, B) is E-stabilizable then the protocol (7) solves the state synchronization problem for any undirected graph $\mathcal{G} \in \mathcal{F}$. Moreover, the synchronized trajectory is given by

$$x_s(t) = A^t \frac{1}{N} \sum\nolimits_{i=1}^{N} x_i(0). \tag{8}$$

and each u_i satisfies the bound (5).

4. SIMULATIONS

Consider the MAS (1) composed by 150 agents and described by

$$A = \begin{bmatrix} 0.4 & 0.8 \\ 0.4 & 0.7 \end{bmatrix};\ B = \begin{bmatrix} -0.6 & 0.002 & -0.2 \\ -0.4 & 0.005 & 0.03 \end{bmatrix};\ E = \begin{bmatrix} 0.07 & 0.2 \\ 0.3 & 0.3 \\ 0.3 & 0.01 \end{bmatrix}$$

with randomly generated initial conditions in the interval $[0, 1]$. Consider also a connected undirected graph $\mathcal{G}$ in the family of regular graphs of degree $d = 7, 20$ denoted by $\mathcal{F}_R \subseteq \mathbb{G}_{[\gamma,\beta]}$. The matrix A is unstable with spectrum $\sigma(A) = \{-0.035235,\ 1.135235\}$.

Let the eigenvalues of every row stochastic matrix $\mathcal{D}$ associated with $\mathcal{G} \subseteq \mathcal{F}_R$ be upper bounded by $\beta = 0.25$ such that $\rho \geq 1/(1-\beta)$, in particular $\rho = 3.83$. To solve the state synchronization problem, the LR-based Protocol 1 is implemented. Consider $s = \mathbb{1} > 0$, and note that

$$\tilde{E} = \frac{1}{\rho}E = \begin{bmatrix} 0.02 & 0.05 \\ 0.07 & 0.07 \\ 0.07 & 0.003 \end{bmatrix}$$

such that $A - |B|\tilde{E}$ is nonnegative. The linear program in Li and Rantzer (2024), is maximized by a vector $p^* = [70.71,\ \ 128.27]$, and results in

$$K = \begin{bmatrix} -0.07 & -0.2 \\ 0.3 & 0.3 \\ -0.3 & -0.01 \end{bmatrix}.$$

Figure 1 shows the evolution of the first and second states for each of the 150 agents in an interconnected system, where each agent is connected to 7 and 20 neighbors, respectively. It is clear from the figure that synchronization is achieved more rapidly as the nodal degree increases. The distance from the trajectories to the synchronized trajectory is represented in Figure (2).

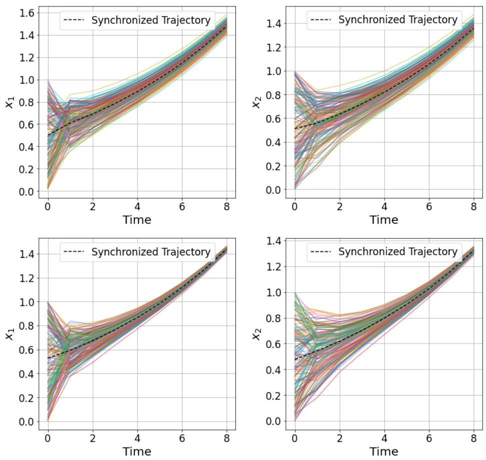

Fig. 1. Evolution over time of the first (left panels) and the second (right panels) state of each agent $i = 1, \ldots, 150$ synchronizing over 7-regular graphs (upper panels) and 20-regular graphs.

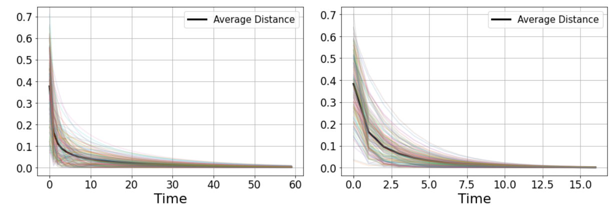

Fig. 2. Euclidean distance to the synchronized trajectory $x_s(t)$ i.e. $\|x_i(t) - x_s(t)\|$ for $i = 1, \ldots 150$ over 7-regular graphs (left panel) and 20-regular graphs.

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