Robust Data-Driven Tube-Based Zonotopic Predictive Control with Closed-Loop Guarantees

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***Abstract*— This work proposes a robust data-driven tube-based zonotopic predictive control (TZPC) approach for discrete-time, linear time-invariant (LTI) systems with unknown dynamics, subject to bounded disturbances and input/state constraints. TZPC is designed to guarantee recursive feasibility and closed-loop stability.**

I. Introduction

Data-driven MPC has been extensively studied, with literature covering both noise-free [1] and noisy scenarios [2], [3]. Recent studies have leveraged zonotope properties to handle unknown but bounded disturbances in both linear [2], [4] and nonlinear systems [5]. Notably, [2] introduced a robust data-driven zonotopic predictive control (ZPC) scheme that ensured robust constraint satisfaction via the propagation of data-driven reachable sets, but lacked guarantees on recursive feasibility and closed-loop stability. To address the latter limitation, [4] introduced a tube-based zonotopic data-driven predictive control (TZDDPC) approach, which achieved closed-loop stability. Nevertheless, the issue of recursive feasibility remained unaddressed.

The proposed TZPC approach addresses both limitations. It ensures recursive feasibility and establishes robust exponential stability of the closed-loop system by integrating suitable terminal ingredients into the control design. Compared to prior methods, TZPC implicitly integrates reachability concepts, leading to notable computational efficiency without compromising robustness. The overall proposed approach consists of two phases: An initial learning phase, which constructs a zonotopic over-approximation of all models consistent with past input and noisy state data; and a control phase, where a robust optimization problem is solved using the model obtained in the learning phase and terminal ingredients to ensure constraint satisfaction, recursive feasibility, and robust stability. This extended abstract provides a summary of the recent results reported in [6].

II. Problem formulation

We consider the discrete-time linear control system

$$x(k+1) = Ax(k) + Bu(k) + w(k), \tag{1}$$

where $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$ are unknown, $x(k) \in \mathbb{R}^{n_x}$ and $u(k) \in \mathbb{R}^{n_u}$, and $w(k) \in \mathbb{R}^{n_x}$ are respectively the state, the control input and a bounded disturbance. The system is subject to state and input constraints:

$$x(k) \in \mathcal{X} \subseteq \mathcal{Z}_x, \quad u(k) \in \mathcal{U} \subseteq \mathcal{Z}_u, \tag{2}$$

where $\mathcal{Z}_x = \langle c_{\mathcal{Z}_x}, G_{\mathcal{Z}_x} \rangle$ and $\mathcal{Z}_u = \langle c_{\mathcal{Z}_u}, G_{\mathcal{Z}_u} \rangle$ denote time-invariant zonotopic sets representing domains of control inputs and states, respectively. Zonotopes are defined as affine transformations of unit hypercubes and take the form $\mathcal{Z} = \langle c, G \rangle$, where c is the center and G is the generator matrix. The disturbance is bounded within a known zonotope: $w(k) \in \mathcal{Z}_w = \langle c_{\mathcal{Z}_w}, G_{\mathcal{Z}_w} \rangle$, with $0 \in \mathcal{Z}_w$, and (A, B) is assumed to be controllable.

The goal is to design a receding horizon optimal controller that guarantees recursive feasibility and robust exponential stability, using only input and state data. We assume access to a input-state trajectory of length $T+1$, and define:

$$\begin{aligned}
X_+ &= \begin{bmatrix} x(-T+1) & x(-T) & \cdots & x(0) \end{bmatrix}, \\
X_- &= \begin{bmatrix} x(-T) & x(-T+1) & \cdots & x(-1) \end{bmatrix}, \\
U_- &= \begin{bmatrix} u(-T) & u(-T+1) & \cdots & u(-1) \end{bmatrix}, \\
W_- &= \begin{bmatrix} w(-T) & w(-T+1) & \cdots & w(-1) \end{bmatrix}.
\end{aligned}$$

We stack the data as $D_- = \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix}^\top$, assuming $\operatorname{rank}(D_-) = n_x + n_u$. Although the disturbance sequence olong the trajectory is unknown, it is captured by a matrix zonotope $W_- \in \mathcal{M}_w = \langle C_{\mathcal{M}_w}, G_{\mathcal{M}_w} \rangle$, obtained through the concatenation of multiple noise zonotopes $\mathcal{Z}_w$.

III. Robust Data-driven Tube-based Predictive Control

This section describes the novel TZPC approach for the unknown linear system (1). This approach consists of an offline learning phase and an online control phase.

Instead of relying on an explicit model, the learning phase leverages measured input-state data to derive a matrix zonotope, a set-valued over-approximation of all possible system matrices consistent with the data and bounded disturbances [2]. Specifically, given input-state data, the set $\mathcal{M}_D = (X_+ \oplus -\mathcal{M}_w) D_-^\dagger$ captures all system models, where $\oplus$ denotes the Minkowski set addition.

As standard MPC, a nominal model $\bar{M} = \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix}$ is chosen from $\mathcal{M}_D$ to define the nominal dynamics:

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k). \tag{3}$$

We assume the existence of a common stabilizing feedback gain K such that the closed-loop nominal system is stable for all models in $\mathcal{M}_D$. The control policy is then defined as:

$$u(k) = \bar{u}(k) + K(x(k) - \bar{x}(k)), \tag{4}$$

Part of this work was supported by the Swedish Energy Authority and IQ Samhällsbyggnad project DOCENT, project number P2023-01513.

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Algorithm 1 Data-driven Tube-Based Zonotopic Predictive Control (TZPC)

Input: Prediction horizon N. Feedback gain K, Constraints $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{X}_N$. Robustly positive invariant set $\mathcal{S}$.

while $t \in \mathbb{Z}_{\geq 0}$ **do**
  Solve the optimal control problem (7).
  Apply $u^*(t) = \bar{u}^*(t|t) + K(x(t) - \bar{x}^*(t|t))$ to the system (1).
  Increase the time step $t = t + 1$.
end while

which ensures that the true system trajectories remain within a bounded tube centered around the nominal trajectory.

The mismatch ΔM between the true and nominal models is rigorously bounded using a zonotopic description:

$$\Delta M \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{Z}_M \oplus \mathcal{Z}_\epsilon, \tag{5}$$

where $\mathcal{Z}_M \oplus \mathcal{Z}_\epsilon$ over-approximate the model mismatch for all $(x, u) \in \mathcal{Z}_x \times \mathcal{Z}_u$. The resulting error dynamics are:

$$e(k+1) = \bar{A}_K e(k) + \phi(k), \tag{6}$$

with $\phi(k) = \Delta M \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + w(k) \in \mathcal{Z}_\phi = \mathcal{Z}_M \oplus \mathcal{Z}_\epsilon \oplus \mathcal{Z}_w$, and $\bar{A}_K = \bar{A} + \bar{B}K$. A robustly positively invariant (RPI) set $\mathcal{S}$ is then computed for the error dynamics (i.e., $\bar{A}_K \mathcal{S} \oplus \mathcal{Z}_\phi \subseteq \mathcal{S}$), ensuring the true state remains within a bounded neighborhood of the nominal state trajectory.

In the online control phase, a robust data-driven predictive control problem is formulated around the nominal system. The optimization considers tightened state and input constraints to ensure robust satisfaction of the original constraints:

$$\min_{\bar{u}_t, \bar{x}_t} \sum_{k=t}^{t+N-1} \ell(\bar{x}(k|t), \bar{u}(k|t)) + \ell_N(\bar{x}(t+N|t)) \tag{7a}$$

$$\text{s.t.} \quad \bar{x}(k+1|t) = \bar{A}\bar{x}(k|t) + \bar{B}\bar{u}(k|t), \tag{7b}$$

$$\bar{u}(k|t) \in \mathcal{U} \ominus K\mathcal{S}, \tag{7c}$$

$$\bar{x}(k|t) \in \mathcal{X} \ominus \mathcal{S}, \tag{7d}$$

$$x(t) \in \bar{x}(t|t) \oplus \mathcal{S}, \tag{7e}$$

$$\bar{x}(t+N|t) \in \mathcal{X}_N \subseteq \mathcal{X} \ominus \mathcal{S}. \tag{7f}$$

Here, $\oplus$ and $\ominus$ denote Minkowski set addition and difference, respectively; ℓ and ℓ_N are the stage and terminal cost functions; and $\mathcal{X}_N$ is the terminal constraint set. The problem is solved in a receding horizon fashion (Algorithm 1), with the optimal solution at time t given by $(\bar{u}_t^*, \bar{x}_t^*)$.

Theorem 1: Under standard terminal conditions (e.g., invariance of $\mathcal{X}_N$ and decrease of ℓ_N under feedback gain K), and assuming that the problem (7) is feasible at initial time $t = 0$, the following closed-loop conditions hold:

(i) The problem (7) is feasible at any $t \in \mathbb{Z}_{\geq 0}$;
(ii) The closed-loop trajectory satisfies the constraints, i.e., $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$, $\forall t \in \mathbb{Z}_{\geq 0}$;
(iii) The set $\mathcal{S}$ is robustly exponentially stable for the resulting closed-loop system.

The proof can be found in [6].

To evaluate the effectiveness of the proposed approach, we consider a benchmark two-dimensional state-space example, used to compare the performance of TZPC against ZPC [2]. The results highlight the proposed controller's ability to maintain robust performance in the presence of external disturbances[1]. Simulation results are shown in Fig. 1. Additionally, we repeated the execution time analysis from [4], reporting maximum solver times of 40 min (ZPC), 78.26 min (TZDDPC), and 0.15 min (TZPC), highlighting its significant computational efficiency.

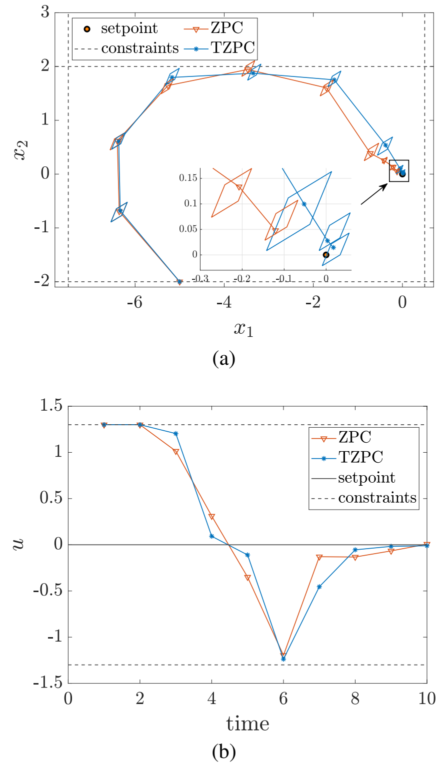

Fig. 1: Comparison between ZPC and TZPC. (a): The reachable sets for the closed-loop system. (b): Optimal control input.

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[1]The example description and corresponding MATLAB code are available at github.com/MahsaFarjadnia/TZPC.