Stability of Open Multi-agent Systems over Dynamic Signed Graphs

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***Abstract*— This paper addresses the bipartite consensus-control problem in open multi-agent systems containing both cooperative and antagonistic interactions. We represent the system as a switched system interconnected over a dynamic signed graph. Using the signed edge-based agreement protocol and constructing strict Lyapunov functions for signed edge-Laplacian matrices with multiple zero eigenvalues, we establish global asymptotic stability of the bipartite consensus control.**

I. Introduction

Open multi-agent systems (OMAS) are networks where agents and edges can dynamically be added to or removed from the system. They naturally arise in applications such as social networks [1], in sensor-based robotic systems [2], and in distributed computation. In most of the literature, agents are assumed to cooperate, but many real-world scenarios involve antagonistic behaviors [3]. A common framework for capturing both cooperation and antagonism is that of *signed graphs*, where edges represent positive (cooperative) or negative (antagonistic) interactions.

In this work, we study the bipartite consensus of OMAS over undirected signed graphs. To the best of our knowledge, this is the first attempt to address this problem. We consider systems where new nodes and edges can be added, and interconnections may switch between cooperation and antagonism. This work reflects real-world scenarios such as social networks and robotic networks. We study OMAS over dynamic signed graphs by modeling them as switched systems [4]. For first-order systems, we establish global asymptotic stability of the bipartite consensus set using Lyapunov's direct method, reformulating the problem in terms of synchronization errors in signed edge coordinates. Building on [5], we extend Lyapunov-based analysis to edge Laplacians with multiple zero eigenvalues. Our main contributions are the construction of strict Lyapunov functions and analysis of how node and edge additions affect convergence.

II. Preliminaries on Signed Graphs

Let $\mathcal{G}_s = (\mathcal{V}, \mathcal{E})$ be a signed graph with node set $\mathcal{V}$ and edge set $\mathcal{E}$. Each edge has a sign: positive (cooperative) or negative (antagonistic). The adjacency matrix $A = [a_{ij}]$ satisfies $a_{ij} \neq 0$ if $(\nu_j, \nu_i) \in \mathcal{E}$, with $a_{ij} > 0$ or $a_{ij} < 0$ depending on the interaction. The graph is *structurally balanced* (SB) if its nodes can be partitioned into two disjoint sets such that intra-group edges are positive and inter-group edges are negative; otherwise, it is *structurally unbalanced* (SUB). The signed incidence matrix $E_s \in \mathbb{R}^{N\times M}$ of $\mathcal{G}_s$, containing N nodes and M edges, is defined as $[E_s]_{ik} = +1$ if ν_i is the initial node of ε_k, $[E_s]_{ik} = -1$ if ν_i, ν_j are cooperative and ν_i is the terminal node of ε_k, $[E_s]_{ik} = +1$ if ν_i, ν_j are competitive and ν_i is the terminal node of ε_k, and $[E_s]_{ik} = 0$ otherwise, where ε_k is the arbitrarily oriented edge interconnecting nodes ν_i and ν_j, $k \leq M,\ i, j \leq N$. The Laplacian matrix $L_s \in \mathbb{R}^{N\times N}$ and the edge Laplacian matrix $L_{e_s} \in \mathbb{R}^{M\times M}$ of an undirected signed graph $\mathcal{G}_s$ can be expressed as $L_s = E_s E_s^\top, L_{e_s} = E_s^\top E_s$.

III. Model and Problem Formulation

Let $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ be the switching signal associated with topology changes, where $\mathcal{P} := \{1, 2, \ldots, s\}$ represents the set of s possible switching modes. Each mode of the system is denoted by $\phi \in \mathcal{P}$, with $\phi = \sigma(\tau)$ for $\tau \in [t_l, t_{l+1})$, where t_l and t_{l+1} are consecutive switching instants.

Assumption 1: The total number of possible switching modes is finite, that is, $\text{card}(\mathcal{P}) < \infty$.

At each mode $\phi \in \mathcal{P}$, the OMAS consists of N_ϕ agents modeled by

$$\dot{x}_i = u_i, \quad i \in \{1, 2, \ldots, N_\phi\}, \tag{1}$$

where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input. The agents interact on a dynamic signed graph $\mathcal{G}_{s_\phi}$ with N_ϕ nodes and M_ϕ edges, and the control law is

$$u_i = -k_1 \sum_{j=1}^{N_\phi} |a_{\phi_{ij}}| \left[x_i - \text{sign}(a_{\phi_{ij}}) x_j\right], \tag{2}$$

where $k_1 > 0$ and $A_\phi = [a_{\phi_{ij}}]$ is the adjacency matrix with $a_{\phi_{ij}} \in \{0, \pm 1\}$.

Assumption 2: The initial signed graph contains a spanning tree.

Under Assumption 2, the achievable control objective for (1) interconnected over an SB graph is to ensure agents achieve *bipartite consensus*, that is, $\lim_{t\to\infty} \left[x_i(t) - \text{sign}(a_{\phi_{ij}}) x_j(t)\right] = 0,\ \forall i, j \leq N_\phi$. If the considered graph is SUB, the achievable control objective under Assumption 2 for (1) is *trivial consensus*, that is, $\lim_{t\to\infty} x_i(t) = 0,\ \forall i \leq N_\phi$. The control objectives for (1) can also be expressed in terms of the synchronization errors, defined as $e_k = x_i - \text{sign}(a_{\phi_{ij}}) x_j, \quad k = (\nu_j, \nu_i) \in \mathcal{E}_\phi$, where $k \leq M_\phi$ denotes the index of the interconnection between the jth and ith agents, and is equivalent to

$$\lim_{t\to\infty} e_k(t) = 0, \quad \forall k \leq M_\phi. \tag{3}$$

This work was supported in part by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg (KAW) Foundation, the Horizon Europe EIC project SymAware (101070802), the ERC LEAFHOUND Project, the Swedish Research Council (VR), and Digital Futures. P. Sekercioglu, A. Fontan, and D. V. Dimarogonas are with the Division of Decision and Control Systems, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden. email: {pelinse,angfon,dimos}@kth.se

IV. Lyapunov Equation for Signed Laplacians

To establish bipartite consensus of OMAS over signed graphs, we construct strict Lyapunov functions in the space of synchronization errors.

Theorem 1: Let $\mathcal{G}_s$ be a signed graph containing N agents interconnected by M edges, and let L_{e_s} be the associated edge Laplacian. If the graph $\mathcal{G}_s$ is a spanning tree, then for any $Q \in \mathbb{R}^{(N-1)\times(N-1)}, Q = Q^\top > 0$, there exists a matrix $P \in \mathbb{R}^{(N-1)\times(N-1)}$, $P = P^\top > 0$ such that

$$PL_{e_s} + L_{e_s}^\top P = Q. \tag{4}$$

Theorem 2: Let $\mathcal{G}_s$ be a signed undirected graph with N nodes and M edges, and let L_{e_s} be the associated edge Laplacian, which contains ξ zero eigenvalues. Then the following are equivalent:

(i) $\mathcal{G}_s$ contains a spanning tree,

(ii) for any $Q \in \mathbb{R}^{M\times M}, Q = Q^\top > 0$ and for any $\{\alpha_1, \alpha_2, \dots, \alpha_\xi\}$ with $\alpha_i > 0$, there exists a matrix $P(\alpha_i) \in \mathbb{R}^{M\times M}$, $P = P^\top > 0$ such that

$$PL_{e_s} = \frac{1}{2}\left[Q - \sum_{i=1}^{\xi} \alpha_i (P v_{r_i} v_{l_i}^\top + v_{l_i} v_{r_i}^\top P)\right], \tag{5}$$

where $v_{r_i}, v_{l_i} \in \mathbb{R}^N$ are, respectively, the right and left eigenvectors of L_{e_s} associated with the ith 0 eigenvalue.

Moreover, if the signed graph is SB, $\xi = M - N + 1$, and $\xi = M - N$ otherwise.

V. Bipartite Consensus on OMAS

At each mode $\phi \in \mathcal{P}$, we have

$$e_\phi = E_{s_\phi}^\top x, \quad u_\phi = -k_1 E_{s_\phi} e_\phi. \tag{6}$$

The closed-loop system for the error dynamics is given as

$$\dot{e}_\phi(t) = -k_1 L_{e_{s_\phi}} e_\phi(t), \quad t \in [t_l, t_{l+1}), \tag{7a}$$

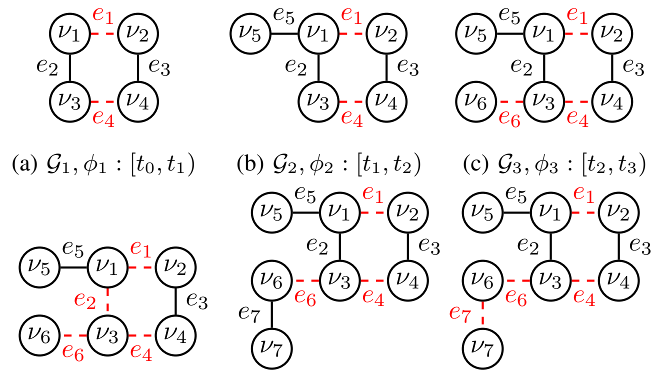
$$e_\phi(t_l^+) = \Xi_{\phi,\hat{\phi}} e_{\hat{\phi}}(t_l^-) + \Phi_l, \quad t = t_l. \tag{7b}$$

Theorem 3: Consider the OMAS (1), under Assumption 2, in closed loop with the switching control law (6). Let $\phi, \hat{\phi} \in \mathcal{P}$ be two consecutive modes.

- Then, if the switching signal σ admits an average dwell time satisfying $\tau_{\phi,\hat{\phi}} \geq \frac{\ln(\Omega_{\phi,\hat{\phi}})}{\gamma_\phi}$, where $\Omega_{\phi,\hat{\phi}}$ and γ_ϕ are positive constants, the origin of the closed-loop system (7a)–(7b) is asymptotically practically stable for all initial conditions.
- Under Assumption 1, the origin of the closed-loop system (7a) is globally asymptotically stable. Furthermore, let $\mathcal{G}_{s_{\bar{\phi}}}$ be the signed graph in the last switching mode $\bar{\phi}$. If $\mathcal{G}_{s_{\bar{\phi}}}$ is SB, then agents achieve bipartite consensus. If $\mathcal{G}_{s_{\bar{\phi}}}$ is SUB, then agents achieve trivial consensus.

VI. Simulation Results

To illustrate our theoretical results, we simulate a system of multi-wheeled mobile robots modeled as unicycles. We apply a feedback linearizing control that redefines each robot's dynamics in terms of the position of a point offset by a fixed distance from the robot's center. After the last switch, the graph $\mathcal{G}_6$ is SB, allowing the agents to achieve bipartite consensus. The peaks on Figure 3 represent the addition of new edges or sign changes in the interconnections.



(a) $\mathcal{G}_1, \phi_1 : [t_0, t_1)$ (b) $\mathcal{G}_2, \phi_2 : [t_1, t_2)$ (c) $\mathcal{G}_3, \phi_3 : [t_2, t_3)$

(d) $\mathcal{G}_4, \phi_4 : [t_3, t_4)$ (e) $\mathcal{G}_5, \phi_5 : [t_4, t_5)$ (f) $\mathcal{G}_6, \phi_6 : [t_5, t_6)$

Fig. 1: Black lines represent cooperative interactions, and dashed red lines represent antagonistic interactions.

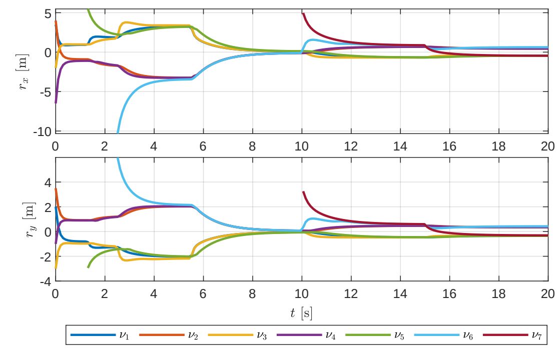


Fig. 2: Evolution of the trajectories of the agents' positions.

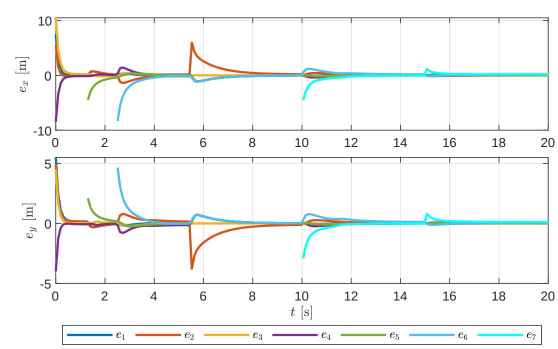


Fig. 3: Evolution of the trajectories of the edges.

References

[1] J. M. Hendrickx and S. Martin, "Open multi-agent systems: Gossiping with deterministic arrivals and departures," in *54th annual allerton conference on communication, control, and computing (Allerton)*, 2016, pp. 1094–1101.

[2] E. Restrepo, A. Loría, I. Sarras, and J. Marzat, "Consensus of open multi-agent systems over dynamic undirected graphs with preserved connectivity and collision avoidance," in *Proc. IEEE CDC*, 2022, pp. 4609–4614.

[3] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE TAC*, vol. 58, no. 4, pp. 935–946, 2013.

[4] M. Xue, Y. Tang, W. Ren, and F. Qian, "Stability of multi-dimensional switched systems with an application to open multi-agent systems," *Automatica*, vol. 146, p. 110644, 2022.

[5] E. Panteley, A. Loría, and S. Sukumar, "Strict Lyapunov functions for consensus under directed connected graphs," in *Proc. European Control Conf.*, St. Petersburg, Russia, 2020, pp. 935–940.